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First passage time distribution of a modified fractional diffusion equation in the semi-infinite interval

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HIGHLIGHTS

- We find the first passage time of a modified fractional equation for accelerating subdiffusion.
- The crossover of the first passage time between two distinct scaling regimes is revealed.
- The scaling behavior is different from that of the aging diffusion.

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ABSTRACT

We investigate the first passage time (FPT) distribution for accelerating subdiffusion governed by the modified fractional diffusion equation which has a secondary fractional time derivative acting on a diffusion operator. For the FPT problem subject to absorbing barrier condition, we obtain exact analytical expressions for the FPT distribution as well as its Laplace transform in the semi-infinite interval. Most of the results have been derived by using the Laplace transform, the Fourier Cosine transform, the Mellin transform and the properties of the Fox *H*-function. In contrast to the Laplace transform of the FPT distribution which can be expressed elegantly and neatly, the exact solution for the FPT distribution requires an infinite series of Fox *H*-functions instead of a single Fox *H*-function. Numerical result reveals that the crossover between the two distinct scaling regimes is apparent only when the discrepancy between the two diffusion exponents becomes more pronounced.

1. Introduction

Diffusion is one of the most important phenomena encountered in numerous physical, chemical and biological systems [1]. However, the picture that has emerged over the last few decades clearly reveals that an increasing number of natural phenomena do not fit into the relatively simple description of normal diffusion [2]. Anomalous diffusion turns out to be quite ubiquitous and normal which is characterized by a nonlinear behavior for the mean square displacement in the course of time [3].

The actual reason or the very nature of anomalous diffusion may vary a lot and there are several approaches or frameworks such as fractional partial differential equations and continuous time random walk models that can be used to describe these anomalous diffusion processes [4]. There are also many physical processes that lack constant power-law

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scaling over the whole time-domain. Such processes including retarding and accelerating anomalous diffusion [5,6] can be described by distributed order fractional diffusion equations [7,8].

More recently, a modified fractional diffusion model has been proposed to describe processes that become less anomalous as time progresses by the inclusion of a secondary fractional time derivative acting on a diffusion operator [9,10],

$$\frac{\partial}{\partial t}P(x,t)=\left(K_\alpha\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}+K_\beta\frac{\partial^{1-\beta}}{\partial t^{1-\beta}}\right)\frac{\partial^2}{\partial x^2}P(x,t), \tag{1}$$

where $0<\alpha<\beta\leq 1$ and the generalized diffusion coefficient K_α, K_β are positive constants with the dimensions $[K_\alpha]=(\text{length})^2\cdot(\text{time})^{-\alpha}$ and $[K_\beta]=(\text{length})^2\cdot(\text{time})^{-\beta}$ respectively.

Note the Riemann–Liouville fractional derivative operator is defined by

$$\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}P(x,t)=\frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t (t-\tau)^{\alpha-1}P(x,\tau)\,d\tau. \tag{2}$$

In fact, there exist different forms of distributed order fractional kinetic equations [7] and even more generalized multi-term fractional diffusion equations in which the fractional derivative may be defined in the Riemann–Liouville or Caputo sense [11–17]. These equations can be used to model a wide range of important physical phenomena including retarding and accelerating subdiffusion and superdiffusion.

So far as the modified fractional time equation in form of Eq. (1) is concerned, it is primarily used to describe the so-called accelerating subdiffusion, introduced and studied in Refs. [7,18]. It is known that such accelerating subdiffusion is characterized by two different scaling behavior of the mean-square displacement, i.e., a smaller short-time power-law exponent and a larger long-time power-law exponent [19]. Among phenomena exhibiting such behavior, one can find the protein motions in cell membranes [20], the diffusion of telomeres in the nucleus of mammalian cells [21], molecules diffusing in living cells [22,23], and a random motion of bright points associated with magnetic fields at the solar photosphere [24]. Another possible application of this equation is in econophysics [25] and particularly the crossover between more and less anomalous behavior has been observed in the volatility of some share prices [26]. In general, the modified equation is advantageous when describing processes which get less subdiffusive in the course of time.

At this point we note that Henry and Wearne [27] find in their derivation of fractional reaction–diffusion equations an additional term

$$\mathscr{L}^{-1}\left[\left(K_\alpha\frac{\partial^{-\alpha}}{\partial t^{-\alpha}}+K_\beta\frac{\partial^{-\beta}}{\partial t^{-\beta}}\right)\frac{\partial^2 P}{\partial x^2}\right]_{t=0}, \tag{3}$$

on the right of Eq. (1) where $\mathscr{L}^{-1}$ is the Laplace inverse transform. The value of this term is unclear as it necessitates the behavior of the term to be known near $t=0$. However it can be shown from the solution of Eq. (1) that these terms are zero and can be neglected [28].

For this modified fractional diffusion equation, Langlands et al. proposed the solution with the form of an infinite series of Fox H-functions on an infinite domain [28]; Liu et al. discussed the numerical method and analytical technique of the modified anomalous subdiffusion equation with a nonlinear source term [29,30]; Chen developed the numerical method to solve the two-dimensional variable-order modified diffusion equation [31].

In connection to these approaches, the first passage time (FPT) plays an important role in the investigation of the stochastic process or the microscopic mechanism of such anomalous diffusion. The FPT distribution gives the probability distribution that a diffusing particle first reaches a specified site at a specified time [32]. Escape times from a random potential, intervals between neural spikes, and fatigue failure times of engineering structures are all examples of FPTs, arising in physics, biology, and engineering, respectively [33]. The first passage probability is mostly concerned with the time required for a stochastically driven particle to first reach a given location and the probability that this location is ever reached [34]. In this context, the knowledge of the FPT distribution is essential for effective probabilistic analysis.

Unfortunately, only in very few cases one has exact analytical expressions for the FPT distribution especially when the anomalous diffusion is considered [35,36]. For the Levy type anomalous diffusion, the FPT distribution or its Laplace transform can be obtained based on the fractional Fokker–Planck equation only for the zero drift case or the non-zero drift case with one absorbing barrier [37,38]. For the usual and fractional nonlinear diffusion equation whose diffusion coefficient is space- and/or time-dependent, analytical solutions can also be given for the FPT distribution in a finite interval and a semi-infinite interval with absorbing barriers [39,40]. The mean FPT and the asymptotic behavior of the FPT distribution for some other cases are also investigated in Refs. [41–43]. Recent theoretical studies associated with first passage phenomena have gone even far beyond pure diffusive transport and/or ideal geometries [34]. With the diverse applications of the first passage theory, the need for analytical FPT distributions becomes more apparent.

For the purpose of this paper, we will only consider the FPT distribution subject to absorbing barrier condition in the semi-infinite interval for the anomalous diffusion process governed by Eq. (1). Our objective is to derive exact analytical solution for the FPT distribution as well as its Laplace transform.