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Enumeration of spanning trees on contact graphs of disk packings

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HIGHLIGHTS

- We investigate the number of spanning trees on contact graph of disk packings.
- An exact analytical expression for this quantity is determined.
- Some electrically equivalent transformations are adopted.
- The new network has small-world scale-free topology with the maximum entropy.

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ABSTRACT

Obtaining the number of spanning trees of complex networks is an outstanding challenge, since traditional approaches, such as calculating the eigenvalues of the matrix and decomposing of spanning subgraphs, are awkward or even infeasible for a large scale network. The foundation and importance of this quantity relating to some topological and dynamic properties prompt us to explore the role of determinant identities for Laplace matrices. We introduce the basic electrically equivalent technique to determine an exact analytical expression for the quantity on the contact graph of disk packings, which is proposed by Zhang et al. (2009). Our theoretical results shed light on the relationship between the microscopic change of the quantity and topological iteration of the network. In particular, we compare the entropy of spanning trees on the network with the other two-dimensional and three-dimensional lattices. We show that the new model is a small-world scale-free network with the maximum entropy so far found. In addition, our method for employing the electrically equivalent technique to enumerate spanning trees is general and can be easily extended to other complex networks.

1. Introduction

With the rapid development of computer technology, there is a considerable progress in acquiring massive data from many real-world systems and even processing them. Recently, many scientists and engineers from various disciplines, such as mathematics, physics, engineering, sociology, and biology, have proposed and investigated a huge variety of network models and inherent mechanisms to mimic topological structures and evolving processes of these networks [1,2]. Despite these research works have led to a significant improvement in the understanding of complexity and characteristics of complex networks, there has been little work dealing explicitly with the geographical effects of all nodes. For most previous studies focusing on topological structures and dynamics of complex networks, the information of spatial or planar location

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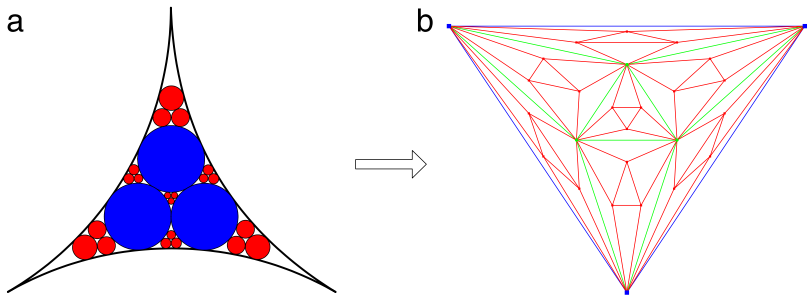

Fig. 1. (a) A variant of Apollonian packing that three disks are added to fill each interstice at arbitrary generation. (b) The corresponding Apollonian network.

of a node is simplified or ignored. However, bus stations in urban transport networks [3], the routers of the Internet [4], and the airports of airline networks [5], are all examples whose nodes have well-defined node positions. It has been shown that such networks associated with geography or spatiality have special topological structures and dynamics [6,7]. In particular, based on space-filling disk packings, Apollonian network is proposed by means of the contact relationships of all disks [8]. Meanwhile, it not only has the features of complex networks such as small-world [9] and scale-free [10], but also has the geographical characteristics of Euclidean and space filling. Accordingly, some research works have been pursued in topological characteristics of high-dimensional Apollonian networks [11,12], multifractal energy spectrum in Apollonian networks [13], correlations in random Apollonian networks [14], as well as contact graphs of disk packings [15].

Unique topological characteristics of Apollonian networks lead a series of interesting research topics. The enumerating spanning trees on a graph or network is one of them, and is also a fundamental issue in mathematics [16–19] and physics [20–22]. The number of spanning trees in a network is generally very large but invariant, even in a small-scale network, which characterizes the reliability of some scale-free unweighted [23,24] and weighted networks [25]. It also determines the mean first-passage time between two nodes in terms of random walks [26]. In addition, it is closed relevant to other interesting problems of graphs or networks, such as synchronization [27] and percolation [28].

The traditional methods to determine this quantity are involved with the calculation of the determinant or the eigenvalues of the Laplacian matrix [29], which is very difficult even intractable for a large-scale network. Recently, it is of considerable current interest to enumerate spanning trees on a scale-free small-world network, since most real networks in nature and society are small-world [9] and scale-free [10]. Based on the self-similarity of network, the spanning trees problem on a pseudofractal scale-free web [30] is firstly considered [31], then the enumeration of spanning trees on an Apollonian network is also investigated [32–34]. This new method associated with the decomposition of spanning subgraphs is effective but complicated for some complex structural networks, since the characteristic subgraphs must be classified with non-repetition and non-omission for all possible subgraphs [34].

In this paper, we investigate the number of spanning trees on a variant of the Apollonian network proposed by Zhang et al. [15], which is devised from the disk packing and the construction method of the Apollonian network, called a contact graph of disk packings. It is constructed whose nodes are described by the circles (generally those nodes contract to the center of the circles) and edges represent contact relation, see Fig. 1. Similar to the Apollonian network, the contact graph of disk packings has the special characteristics in nature and society: large clustering coefficients, small-world effect and a power-law degree distribution[15]. We employ techniques from the theory of electrical networks [19] – such as Kirchhoff's current law and some simple transformations – to determine the number of spanning trees of this variant, and compute the entropy of its spanning trees.

2. Preliminaries

Let G be a simple connected network with node set $V(G) = \{v_1, \ldots, v_n\}$. A spanning tree of the network is a minimal set of edges that connect all nodes. The degree of node v, denoted by $d(v)$, is the number of edges attached to it. $D(G)$ denotes the diagonal matrix $(d_{ij})_{n\times n}$ whose elements are $d_{ii} = d(v_i)$. The adjacency matrix $A(G)$ of G is a matrix $(a_{ij})_{n\times n}$ whose elements are $a_{ij} = 1$ if nodes v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. Then the Laplacian matrix $L(G)$ of G is the matrix $L(G) := D(G) - A(G)$. The reader is referred to Biggs [29] for these and further related definitions.

It should be pointed out that two typical methods related to the Laplacian matrix for enumerating spanning trees on G. One method is "the Matrix-Theorem" [29], which expresses the number of spanning trees $t(G)$ of G as a determinant: $t(G)$ equals the determinant of the submatrix obtained by deleting row v_r and column v_r from $L(G)$ for any $1 \le r \le n$. The other is "the eigenvalue method" [29]. Let $\lambda_1(G) \ge \lambda_2(G) \ge \cdots \ge \lambda_n(G)$ denote the eigenvalues of $L(G)$. Then it is easily shown that $\lambda_n(G) = 0$. Furthermore, $t(G) = \frac{1}{n}\prod_{i=1}^{n-1}\lambda_i(G)$. If $t(G)$ grows exponentially with $|V(G)| = n$ as $n \to \infty$, there exists a constant $E(G)$, called the entropy of spanning trees of G, describing this exponential growth [17,18]:

$$E(G) := \lim_{n\to\infty} \frac{\ln t(G)}{n}. \quad (1)$$