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A closed solution to the Fokker–Planck equation applied to forecasting

Chris Montagnon

Department of Mathematics, Imperial College, London, SW7 2AZ, United Kingdom

HIGHLIGHTS

- Closed solutions are obtained to the Fokker–Planck equation with linear and with quadratic diffusion terms.
- Applied to real world time series to generate probability distribution forecasts.
- Parameters in the models are dynamically estimated from past data.
- Solutions compared to those from autoregressive models with good results.

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ABSTRACT

Closed solutions to the Fokker–Planck equation with linear and quadratic diffusion terms have been generated. These are applied to forecasting time series to generate a probability distribution for the next point, rather than a single point estimate as in autoregression. The parameters in these models are dynamically estimated from past data. The resulting solutions are compared to a conventional autoregressive approach with encouraging results.

1. Introduction

In forecasting a time series, $\{X_t, t = 1, \ldots, N\}$, for many real world situations, rather than a forecast of the 'best' estimate for the next point X_{N+1}, what is needed is a probability distribution of the possible values of X_{N+1}. Kantz and Schreiber [1] mentioned tackling this through the use of a Fokker–Planck equation in a probability distribution function $W(x, N+t)$, where $W(x, N+t).dx$ is the probability of finding the actual X_{N+t} in $(x, x+dx)$ at time t after N, but they do not take this further because of difficulties in estimating the parameters. Several more recent papers (e.g. Refs. [2–4]) have sought to use forecasting methods based upon a non linear Langevin equation leading to a Fokker–Planck equation, but any solutions have been numerical. Some, e.g. Ref. [5], estimate the moments $M^{(n)}(x) = \langle (X_{t+1} - x)^n \rangle|_{X(t)=x}$ from the data and thus create the drift and diffusion coefficients but they also solve the resulting Fokker–Planck equation numerically. Furthermore sampling intervals and noise can cause distortions in these estimates of the moments [6,7]. Thus the two main problems in using a Fokker–Planck equation for forecasting are: finding the drift and diffusion coefficients and solving the resulting equation.

In this paper we generate a closed analytical solution to the Fokker–Planck equation for various polynomials for these drift and diffusion coefficients, which enables us to choose the parameters for these polynomials by selecting values that generate the 'tightest' (i.e smallest variance) distribution that meets the probabilistic criteria of the past data.

E-mail address: chrismontagnon@compuserve.com.

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In Section 2 we lay out three models for the drift and diffusion coefficients: a model with linear drift and constant diffusion (an Ornstein Uhlenbeck process); a model with linear drift and linear diffusion; a model with linear drift but quadratic diffusion. We indicate how to generate a closed solution to the Fokker–Planck (FP) equation in each case. In the last section we report on computational experience applying these results to some time series (daily sales data from a supermarket), that seem to be quite volatile [8], and compare these to results from using a standard autoregressive (ARMA) approach.

2. The models

The time series of measurable quantities can be modelled by means of a linear autoregressive moving average model (ARMA) and also by a non-linear Langevin equation with the latter leading to a Fokker–Planck (FP) equation in $W(x, \tau + t)$, where $W(x, \tau + t).dx$ is the probability of the actual $X_{\tau+t}$ being in the interval $(x, x + dx)$ and where τ is the time of the last known observation. The resulting Fokker–Planck equation is:

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x}(D^{(1)}.W) + \frac{\partial^2}{\partial x^2}(D^{(2)}.W)$$

where $D^{(1)}$ is the drift coefficient and $D^{(2)}$ is the diffusion coefficient. We solve this for different models for $D^{(1)}$ and $D^{(2)}$.

This equation is solved for $t > 0$ for each τ with boundary condition $W(x, \tau + 0) = \delta(x - X_\tau)$, where X_τ is the actual observation at time τ. We confirm that the solution has $\int_{-\infty}^{+\infty} W(x, \tau + t).dx = 1$ for all t as $W(x, \tau + t)$ has to remain a probability distribution. Thus we have an ensemble of solutions, one at each τ giving, for that particular τ, a family of distributions for each of the outcomes at time given by $t = 1, 2, 3, \ldots$.

When we compare results for these ARMA and Fokker–Planck models we should note that they differ in both objective and assumptions:

The objective of an ARMA model is to forecast as accurately as possible the next actual value, i.e. $\hat{X}_{\tau+t}$: any distribution for the actual $X_{\tau+t}$ is taken as Gaussian with mean $\hat{X}_{\tau+t}$ and a variance which has been estimated using past experience of the inaccuracy of these forecasts. For the Fokker–Planck model the objective from the outset is to predict, not a particular value, but the whole distribution of the possible values $X_{\tau+t}$: i.e. the probability distribution $W(x, \tau+t)$ where $W(x, \tau) = \delta(x-X_\tau)$.

These ARMA and Fokker–Planck models are also based upon very different assumptions. For (p, q) with $p > 1$ ARMA assumes the time series not to be Markovian, to be stationary and ergodic. The Fokker–Planck model assumes that the series is Markovian and it does not need the stationarity or the ergodicity assumptions. Thus for the Fokker–Planck solution we can adapt the parameters based upon the current situation at time τ.

2.1. Constant diffusion

In all our models we take the drift coefficient $D^{(1)} = -\gamma x$. In this first model we take the diffusion coefficient to be constant: $D^{(2)} = c$

The resulting Fokker–Planck equation is:

$$\frac{\partial W}{\partial t} = +\frac{\partial}{\partial x}(\gamma.x.W) + c.\frac{\partial^2}{\partial x^2}W \ldots. \tag{2.1}$$

The solution to this is well established (e.g. see [9]) as:

$$W(x, \tau + t) = \sqrt{\left(\frac{\gamma}{2\pi c(1 - e^{-2\gamma t})}\right)}.\exp\left[-\frac{\gamma.(x - e^{-\gamma t}x_\tau)^2}{2c(1 - e^{-2\gamma t})}\right]\ldots. \tag{2.2}$$

2.2. Linear diffusion

In this case we choose the diffusion coefficient to be $D^{(2)} = c + bx$, but we keep the drift coefficient as $D^{(1)} = -\gamma.x$. This generates a FP equation:

$$\frac{\partial W}{\partial t} = +\gamma W + (\gamma x + 2b).\frac{\partial}{\partial x}(W) + (c + bx).\frac{\partial^2}{\partial x^2}W \ldots. \tag{2.3}$$

To solve this we take a Fourier transform to give:

$$\frac{\partial \hat{W}}{\partial t} = -ck^2.\hat{W} - (\gamma k - ibk^2).\frac{\partial}{\partial k}.\hat{W} \ldots. \tag{2.4}$$

We solve for $\hat{W}$ by the method of characteristics with the initial condition $\hat{W}(k, \tau + 0) = e^{-i.kX_\tau}$. However to complete the reverse FT we have to show that $W(x, \tau + t)$ is equal to the residues inside a contour integral along the path PQR in the complex plane, where PQ is a straight line from $P = (1, -b'm(1 - e^{-\gamma t})$ to $Q = (1, +b'm(1 - e^{-\gamma t})$ and QRP is an arc of radius $= \sqrt{1 + (b')^2m^2(1 - e^{-\gamma t})^2}$. (Note that we have defined $c' = c/\gamma$ and $b' = b/\gamma$)

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