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# The role of boundary on equilibrium configuration of rotational symmetric gel sheets

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## HIGHLIGHTS

- The conformation mechanism of equilibrium gel shell is reformed by the boundary.
- The 90% stretching energy is found accumulating on the outer 10% part of sheet.
- A simple method to measure boundary line tension is proposed.

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## ABSTRACT

In rotational symmetric gel sheets, we study the sheets with in-plane strain and deduce the equilibrium equations and the boundary conditions. The boundary is found to play an important role in the equilibrium configuration of the gel sheet. Compared to the sheet without a boundary, with a boundary the strain enlarges extremely. The domination between bending and stretching can be changed by the boundary. In gel sheets with a boundary, the forming mechanism of equilibrium configuration is different from the sheet without a boundary. The equilibrium configuration is governed by the cooperation of boundary, bending, and stretching rather than just the last two. Furthermore, we find that the boundary line tension $\gamma$ cannot be omitted. If $\gamma$ is ignored, the elastic energy roughly enlarges by 5%. Finally, we discuss the effects of $\gamma$ and propose a way to measure it by the border radius.


## 1. Introduction

The thin biological tissues can be regarded as sheets, such as cell membrane, skin, flowers, and leaves. The shape of these sheets determines their biological function. Thus, understanding the configuration mechanism of the shape of these sheets is significant. It is found that the sheets have different shapes depending on the gene [1–8] and external environment [9–11]. E. Sharon et al. point out that these factors can be described by the intrinsic geometries [1,4]. They provide a simple model for studying the leaves by gel sheets. The sheets, which have nonuniform shrinkage, can form different configurations, for example, the rotational symmetric sheets have a brim-like boundary [12,13], while the nonrotational symmetric sheets are wave like [12–16].

Different from the sheet without a boundary, the sheets with a boundary have special shapes. It has been found that the curving direction of the surface near the boundary is different from the center part, such as leaves with wavy edges [1], the leaves of Acetabularia schenckii [5], and the vesicle with holes [16]. The Gaussian curvature of the surface is negative near the boundary, which can also be verified by theoretical studies [17–21]. For the sheet with elliptic metrics, one finds

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that the sheet has abundant stretching energy near the boundary [21]. This discovery implies that the boundary has a special mechanical characteristic. However, for the general sheets with in-plane strain, it is unknown whether the stretching energy is always cumulated near the boundary. Furthermore, whether the boundary can affect the forming mechanism of equilibrium configuration is also unclear.

In this paper, we investigate the role of the boundary in the equilibrium configuration in rotational symmetric gel sheets. In addition, we study how the boundary imposes an adjustment upon the equilibrium configuration. First, we discuss the equilibrium configuration in different cases to study the role of the boundary and find the origin of boundary characteristics. Second, we study the sheet with and without a boundary. The boundary increases the in-plane strain obviously. It plays a crucial role in the gel configuration. The boundary, bending, and stretching together govern the equilibrium shape of the gel sheets. Finally, we show the effect of line tension $\gamma$ and propose a simple method to measure the boundary line tension.

This paper is organized as follows: In Section 2, for rotational symmetric sheets, the equilibrium equations and corresponding boundary conditions are derived. In Section 3, by comparing the sheets with and without a boundary, we study the physical effects of the boundary. Moreover, we discuss how the concentration, thickness, and Poisson ratio influence the boundary characters in Section 4. In Section 5, the relationship between the equilibrium configuration and the boundary line tension is deduced. Section 6 includes a short conclusion and remarks.

## 2. The equilibrium shape equations and boundary conditions

In the gel sheets, we define two surface states, named as target state $\tilde{S}$ and equilibrium state $S$. The target state $\tilde{S}$ is an ideal configuration with no in-plane strain. The equilibrium state $S$ is the final stable configuration of gel sheets.

The thin gel sheets can be represented as a two-dimensional (2D) curved surface. For thin sheets, the elastic energy can be written as $E = E_s + E_b$. The bending energy $E_b$ is denoted as the Helfrich form [22]. The in-plane stretching energy $E_s$ is represented by the displacement 2D vector field $\mathbf{u}$ on the surface of the sheet which is determined by the difference between equilibrium state $S$ and target state $\tilde{S}$ [18]:

$$\begin{aligned} E_s &= \int_{\tilde{S}} w_s d\tilde{A}, \\ w_s &= \frac{1}{2}(2\mu u_{\alpha\beta}^2 + \lambda u_{\alpha\alpha}^2), \\ E_s &= \int_S w_b dA, \\ w_b &= \frac{\kappa}{2}H^2 + \kappa_G K, \end{aligned} \tag{1}$$

where $w_s$ and $w_b$ are the energy densities of in-plane stretching energy and bending energy, and $d\tilde{A}$ and $dA$ are the area elements on the surface $\tilde{S}$ and $S$, respectively. $\mu$ and $\lambda$ are the 2D Lamé coefficients. $H$ is the mean curvature (we adopt it as the sum of principal curvatures rather than the average). $K$ is the Gaussian curvature. $\kappa$ and $\kappa_G$ are the bending rigidity and Gaussian rigidity, respectively.

For the gel sheet, the molecules on both sides of boundary $C$ are different. The gel's molecule is inside, and the environmental molecule is outside. Thus, the boundary suffers a residual force which induces an additional energy on it. We can define a boundary line tension $\gamma$, which represents the line density of additional energy on the boundary. This additional energy can be written as $E_C = \oint_C \gamma ds$ [23,24]. In the gel sheet, the shape of the target state is determined by the initial concentration, so the value $\gamma ds$ of $\oint_C \gamma d\tilde{s}$ is constant. It can be regarded as a referenced potential energy. Then, the boundary energy term of the gel sheets is

$$E_C = \oint_C \gamma ds - \oint_C \gamma d\tilde{s} \tag{2}$$

where $ds$ and $ds$ are the line elements on the boundary of the equilibrium state and the target state, respectively.

Then, the configuration energy can be written as [25]

$$\mathcal{H} = E_s + E_b + E_C. \tag{3}$$

For the surfaces $\tilde{S}$ and $S$ of the rotational symmetric sheet, we choose cylindrical coordinates $(\tilde{\rho}, \varphi, \tilde{z}(\tilde{\rho}, \varphi))$ and $(\rho, \varphi, z(\rho, \varphi))$ to describe them, respectively. The in-plane strain tensor $\mathbf{u}$ is [18,26]

$$\begin{aligned} u_{11} &= \frac{1}{2}\left[\left(\frac{\rho}{\tilde{\rho}}\right)^2 - 1\right], \\ u_{22} &= \frac{1}{2}\left[\left(\frac{dl}{d\tilde{l}}\right)^2 - 1\right], \\ u_{12} &= u_{21} = 0 \end{aligned} \tag{4}$$

where $\tilde{l} = \int \sqrt{1 + \tilde{z}_{\tilde{\rho}}^2} d\tilde{\rho}$ and $l = \int \sqrt{1 + z_\rho^2} d\rho$.