



Analytical solutions for a nonlinear diffusion equation with convection and reaction



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HIGHLIGHTS

- We discuss solutions of a nonlinear diffusion equation.
- An analytical technique based on a power-law ansatz is developed.
- We analyze cases involving the reaction and convection.

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ABSTRACT

Nonlinear diffusion equations with the convection and reaction terms are solved by using a power-law ansatz. This kind of equations typically appears in nonlinear problems of heat and mass transfer and flows in porous media. The solutions that we introduce in this work are analytical. At least, in the convection case, the result recovers its linear form as a special limit. In the reaction case, we define a class of nonlinearity to discuss the evolution of general solutions, we also add the Verhulst-like dynamics and global regulation. We think this method, based on this kind of ansatz, can also be applied to solve other nonlinear partial differential equations.

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1. Introduction

Particular attention has been paid to solutions of partial differential equations. Diffusion processes and related problems in physics and complex systems are invoked in some textbooks [1–5]. Many specialists in different areas of knowledge have made great efforts to write simple but exact solutions of nonlinear differential equations related to their own interests using computer symbolic systems, as Maple or Mathematica softwares, that help to perform complicated and tedious algebraic calculations on computers.

Years ago, it was believed that the diffusion equation had fully described many problems related to the physics, engineering and other topics of the exact sciences and complex systems. Particularly, diffusive problems were considered linear [5] but the increase of the knowledge related to the transfer of information, and the applications to diverse systems, increased the complexity, considering that the loss of the predictability leads to a nonlinear response [6–11,1]. In this line of the research, we show two examples of nonlinear partial differential equations, which can be exactly solved through an ansatz of power law that matches with an exponential function as a limiting case.

The general nonlinear diffusion equation is given by

$$\frac{\partial W(x, t)}{\partial t} = \frac{\kappa}{1 + \alpha} \frac{\partial^2 W(x, t)^{1+\alpha}}{\partial x^2}. \quad (1)$$

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This family of equations is classified in the literature as porous media equation and a possible application is considered in the [Appendix](#).

The convection and source terms can be added into (1) to obtain a general form of the nonlinear diffusion equation, which is also identified [12] as quasi-linear equation, is given by

$$\frac{\partial}{\partial t} W(x, t) = \frac{k}{1 + \alpha} \frac{\partial^2}{\partial x^2} W^{1+\alpha}(x, t) + g(x)W^v + f(x)W^s(x, t) \frac{\partial}{\partial x} W(x, t). \quad (2)$$

The case, where $f(x)$ and $g(x)$ are constant, has been already studied [12]. In relation to the previous equation, at the right side, the second term describes the volumetric absorption and the third term emphasizes the convective nature of the systems. Thus, the primary goal of this work is to derive solutions for the partial differential equation (2).

Some of the relevant applications, of nonlinear diffusion equations that are found in the literature, are highlighted:

Phonon transport [13]: The phonon transport within silicon structures, which is subjected to internal heat generation can be explored. Results indicated that the linear heat diffusion equation significantly underestimates temperature distribution at nanoscales in the presence of an external heat source. Discussions are provided on temperature distribution inside silicon thin film when heated by a pulsed laser, an electron beam or due to near-field thermal radiation effects. Nonlinear heat diffusion models ([Appendix](#)) are strong candidates to explain the problem at nanoscales with heat generation.

Biology [14]: Mathematical biology is a fast-growing and well-recognized subject that corresponds to the most exciting modern application of mathematics. The increasing use of mathematics in Biology is inevitable as Biology becomes more quantitative. The complexity of the biological sciences makes interdisciplinary involvement essential. For the mathematician, Biology brings out new and exciting branches.

Archeology [15]: A nonlinear mathematical model for the Trojan war that took place around 1180 BC is supported by archeological findings and by Homer Iliad. The number of warriors, the struggle rate parameters, the number of individuals per hectare, and other related quantities are estimated. The long siege of the city, described in the Iliad, is compatible with a power law behavior for the time evolution of the number of individuals.

Galactic civilizations [16]: The interstellar diffusion of galactic civilizations is studied by potential theory; both numerical and analytical solutions are derived for the nonlinear partial differential, and difference equations which specify a range of relevant models drawn from blast wave physics, soil science, and especially population biology, etc.

The analytical solutions constructed in the present study correspond to a family of power laws, that can be connected with the q -exponential functions of the Tsallis formalism of the statistical mechanics [17]. As made before [7–10], taking into account the following transformation $\alpha = 1 - q$ into Eq. (1) or (2), we obtain a family of nonlinear differential equation that is extensively studied in recent literature. The linear diffusion equation and its solutions are undoubtedly recovered when the limit $q \rightarrow 1$ is considered.

The rest of the paper contains the following topics: in Section 2, the nonlinear diffusion equation with convection is discussed. In Section 3, the nonlinear diffusion equation with reaction is studied. Finally, we draw some concluding remarks in Section 4. Additionally, a direct derivation of Eq. (1) is introduced through a possible application in the [Appendix](#).

2. Nonlinear diffusion equation with convection

In this section, we study Eq. (2) with $s = 0$, $g(x) = 0$ and $f(x) = -v$; and, as indicated above, with $\alpha = 1 - q$ expressing the nonlinearity by the parameter q . This nonlinear differential equation, which is a family of nonlinear heat equations with convection, is presented as follows:

$$\frac{\partial W(x, t)}{\partial t} = \frac{\kappa}{2 - q} \frac{\partial^2 W(x, t)^{2-q}}{\partial x^2} - v \frac{\partial W(x, t)}{\partial x}, \quad (3)$$

where v is the velocity. A pertinent is that, the function $W(x, t)$ is dimensionless to avoid an inappropriate use of physical quantities involved in the nonlinear partial differential equation (3). For instance, if $W(x, t)$ represents the temperature, then $W(x, t) = T(x, t)/T_s$, where T_s is constant and represents a scaling factor for the temperature. Other property of the function $W(x, t)$ comes from integrating Eq. (3) over all the space, to obtain $\int_{\text{all space}} dx \partial W(x, t) / \partial t = 0$, this means that the quantity $\int_{\text{all space}} dx W(x, t)$, which corresponds to the total heat, is constant.

As alluded before, a possible application of this equation is the phonon transport within silicon structures at nanoscales with heat generation [13].

2.1. Power law diffusion type solution

Now, we solve exactly Eq. (3) using a power law similar to the ansatz discussed by Plastino and Plastino [8]. The main idea is to introduce the ansatz into the nonlinear equation (3) to obtain a set of coupled differential equations. Thus, we

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