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A unified characterization of generalized information and certainty measures☆

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HIGHLIGHTS

- The bounds of the Inforcer measure are derived, which generalizes a previous result.
- The boundness of information and certainty measures generated by the Lorentz velocity is discussed.
- The intuitive justification of the axiomatic systems is given.
- A possible application of our generalized measures for the characterization of system statistical complexity is discussed.

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ABSTRACT

In this paper we consider the axiomatic characterization of information and certainty measures in a unified way. We present the general axiomatic system which captures the common properties of a large number of the measures previously considered by numerous authors. We provide the corresponding characterization theorems and define a new generalized measure called the Inforcer, which is the quasi-linear mean of the function associated with the event probability following the general composition law. In particular, we pay attention to the polynomial composition and the corresponding polynomially composable Inforcer measure. The most common measures appearing in literature can be obtained by specific choice of parameters appearing in our generic measure and they are listed in tables.

1. Introduction

In the past decades there was a plausible interest for the definition and characterization of the measures of information and certainty associated with a probability distribution. The most commonly used ones are those which can be obtained as the average value of the information/certainty associated with the event.

Information measures determine the amount of uncertainty associated with a probability distribution. The basic one is the Shannon entropy [1], defined as a linear (trace-form) expectation of an additive decreasing function of an event probability called information content. Renyi [2], Varma [3] and Nath [4] considered the class of entropies which can be obtained as a quasi-linear mean weighted by the random variable probability. Entropies with more general weights were considered

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by Aczél and Daróczy [5], Kapur [6], Rathie [7], Khan and Autar [8] and Singh et al. [9]. Havrda and Charvát [10] and, subsequently, Daróczy [11] and Tsallis [12], considered the entropies which are the trace form of pseudo-additive information content. The trace form entropies based on the pseudo-additive content are also considered by Abe [13] and Kaniadakis [14,15]. The class of entropies which are a quasi-linear mean of the pseudo-additive information are considered by Sharma and Mittal [16], Frank and Daffertshofer [17], Arimoto [18], Boekee, Boxma and Van der Lubbe [19,20] and Picard [21].

Inaccuracy measures are a generalization of entropies, which deal with two distributions and reduce to the entropies if the distributions are identical. The first one introduced was the Kerridge inaccuracy [22], defined as the expected value of the information content of the first distribution, where the weights are event probabilities with respect to the second distribution. Nath [23] considered two types of the generalization—one to quasi-linear means and the other with pseudo-additive information content. The combination of these two approaches was considered by Gupta and Sharma [24].

Certainty measures are defined as the average value of a multiplicative increasing function of the event probability called the certainty content. The certainty measures which can be represented as the trace form expectation of the certainty content are Onicescu's information energy [25], also called Weaver's expected commonness [26], and order-α weighted information energy introduced by Pardo [27]. The certainty measures which can be represented as the quasi-linear expectation were considered by Van der Lubbe [20] and Bhatia [28].

In this paper we propose the general axiomatic system, which characterizes in a unique manner the majority of the known information and certainty measures, and obtain the Inforcer measure as the unique one that satisfies it. By our axiomatic system, information and certainty measures are the particular cases of the Inforcer measure, which is the quasi-linear mean of the Inforcer content. The Inforcer content is defined as a monotonic function of event probability, having the information and certainty content as special cases. The Inforcer measure and the Inforcer content follow the simple composition rule, which asserts that the Inforcer measure and the Inforcer content of joint distributions can be obtained as the composition of the Inforcer measure and the Inforcer content of particular distributions.

In particular, we pay attention to the most common measures appearing in literature, generated by the polynomial composition operation. The polynomial composition has already been considered by Behara and Nath [29] and by Ebanks [30] for the case of trace form entropies. Instead of this, we derive the more general form for information and certainty measures generated as a quasi-linear mean of the content which follows the polynomial composition law. It is shown that pseudo-addition [31] and real product represent the unique polynomial composition operations which preserve the decreasing information and increasing certainty content. Finally, we provide the theorem which represents the interplay between the polynomial certainty and information measures, generalizing the result from Ref. [20].

The general axiomatic system for the Inforcer measure and the corresponding uniqueness theorem are presented in Section 2. In Section 3 we define the information and certainty measures as instances of the Inforcer measure and consider their boundedness, generalizing the result from Ref. [32]. In Section 4 we consider polynomial composition, derive the corresponding information and certainty measures and briefly consider their physical properties. It is shown that the majority of the information and certainty measures previously considered in literature can be obtained by instantiation of the Inforcer measure. The measures are listed in Tables 1 and 2. In Section 5 we establish the connection between information and certainty measures, by which an information measure is uniquely determined as a decreasing function of a certainty measure, generalizing the result from Ref. [20].

2. Axiomatic characterization of the Inforcer measure

Let $h : \mathbb{R} \to \mathbb{R}$ be a monotonic continuous (hence invertible) function such that $h(x) > 0$ for $x > 0$ and let the composition operation $\odot$ be defined as:

$$h(a + b) = h(a) \odot h(b); \quad a, b \in \mathbb{R}. \tag{1}$$

Let the set of all n-dimensional distributions and the set of all the positive ones be denoted with

$$\Delta_n \equiv \left\{ (p_1, \ldots, p_n) \middle| \, p_i \geq 0, \sum_{i=1}^{n} p_i = 1 \right\}; \qquad \Delta_n^+ \equiv \left\{ (p_1, \ldots, p_n) \middle| \, p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\}; \quad n > 1, \tag{2}$$

respectively. Let the *direct product*, $P \star Q \in \Delta_{nm}$, be defined as

$$P \star Q = (p_1 q_1, p_1 q_2, \ldots, p_n q_m), \tag{3}$$

for $P = (p_1, \ldots, p_n) \in \Delta_n$ and $Q = (q_1, \ldots, q_m) \in \Delta_m$ and let $\mathbb{R}^+$ denote the set of positive real numbers.

The Inforcer measure is characterized with the following set of axioms.

[A1] The *Inforcer content* $\mathcal{EC} : (0, 1] \to \mathbb{R}^+$ is a continuous monotonic function, which is composable:

$$\mathcal{EC}(pq) = \mathcal{EC}(p) \odot \mathcal{EC}(q), \quad \text{for all } p, q \in (0, 1]. \tag{4}$$