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Synchronization of Kuramoto oscillators in small-world networks

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HIGHLIGHTS

- We analyze influences of several key parameters on phase synchronization of Kuramoto oscillators in small-world networks.
- We discover a new phenomenon that too large coupling strength is not conducive to the phase synchronization.
- We present a simple model to promote the synchronization performance in strong coupled system.

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ABSTRACT

We study the synchronization of Kuramoto oscillators on small-world networks. Firstly, we analyze the impact of coupling strength, rewiring probability and average degree of networks on the phase synchronization in weak coupled system. Then, by increasing the coupling strength, we find an interesting phenomenon that when coupling strength is much larger than the synchronization threshold, there is another threshold which can decrease the synchronization performance of the system. Finally, in order to enhance the synchronization performance of strong coupled system, we adopt a simple way to delete some nodes randomly to construct disordered environment, numerical results show that a certain degree of disorder can promote the synchronization performance of strong coupled small-world networks.

1. Introduction

From synchronous flashing of fireflies [1] to applause in music hall [2], from chorusing crickets [3] to coupled laser arrays [4], the synchronous behaviors are not only widespread in the biological world, but also in society. In recent decades, scholars try to find the necessary conditions and mechanisms that result in synchronization among self-organizational group [4–6]. Probably the earliest and the most inspiring model was suggested by Kuramoto [7], who shows that there is a coupling strength threshold which can lead to synchronization of a set of oscillators with fixed amplitude mutually coupled by a 2π periodic interaction. Science then, various extensions and generalizations models based on Kuramoto model have been analyzed more deeply [5,6,8,9]. In recent years, especially, after the introduction of complex network [10,11], studies of synchronization have mostly been transferred to the influences of structural properties of small-world (SW) [12–14], as well as scale-free (SF) network [15,16]. The earlier studies of synchronization of Kuramoto oscillators on top of SW networks and SF network are in Refs. [12] and [15] respectively, and then, Hong et al. studied the factors that can promote synchronization

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on complex networks [17,18]. Along this path, many studies which mainly focus on how to improve synchronization of Kuramoto oscillators on complex network have been advanced [19–23], which correspond to the main direction of the study on synchronization of complex network.

However, in the process of pursuing mainstream of research, some initial problems may be slighted. In fact, one of the most important and initial problems in Kuramoto model is whether there is a threshold value K_c which can emerge the synchronization of coupled oscillators. Of course, Kuramoto [7] and many other scholars have conducted discussions on varied versions of this topic, which have been comprehensively reviewed in Refs. [9] and [24]. It has been found that when there is a threshold K_c, the phase synchronization of coupled oscillators will emerge, for example, when the coupling strength $K > K_c$, the system will maintain partial synchronization, when K is large enough, specially, when $K \to \infty$, complete synchronization will emerge. However, previous studies are almost based on the regular networks [5,6] or complete graph [7,25] rather than SW networks. Only in a few studies [12,13,23] in recent years, on the other hand, the Kuramoto coupled oscillators has been considered on SW networks and the system synchronization has been investigated. It is a pity that on matter in analytical or numerical studies, researchers focus their efforts on finding out the threshold K or investigating the system dynamical mechanism when K approaches K_c, but ignore the case when K is much larger than K_c.

Different from previous researches that only focus on the case of K approaching K_c (weak coupled, the corresponding system is called weak coupled system), in this paper, we numerically investigate the dynamical system of Kuramoto coupled oscillators on SW networks for the case of $K \gg K_c$ (strong coupled, the corresponding system is called strong coupled system). In contrast to the existing results, we find an interesting phenomenon that there is another threshold K_d in strong coupled system. When $K > K_d \gg K_c$, the synchronization performance of the dynamical system will decrease with the increase of K. For another important problem that how to improve the synchronization in strong coupled system on SW networks, we develop a simple model which the network nodes are randomly deleted (diluted networks). Numerical results show that this simple method can promote the synchronization performance of the strong coupled system effectively.

The paper is structured as follows. In the next section, we present the Kuramoto model on SW networks and the parameters to be measured. Then we compare the numerical results of weak coupled system and strong coupled system and try to analyze the reason of results. A simple model which can promote the synchronization performance of the strong coupled system is proposed and the numerical results are showed in next section. Finally, we draw the conclusions with a brief discussion in the last section.

2. Kuramoto model on SW networks

Kuramoto model is undoubtedly one of the simplest and the most successful models in exploring the synchronization of coupled oscillators. In this section, we introduce the Kuramoto model on the underlying topology of a SW networks. The SW network considered in this paper is constructed in the following way [10]: First, a one-dimensional regular network of N nodes under periodic boundary conditions is constructed, with only nearest neighbor connections of range $k/2$ between the nodes. Then each local link is visited once, it is removed and reconnected to a randomly chosen node with the rewiring probability p. Multi-connections and self-connections are prohibited during the process.

At each node of this SW networks, an oscillator is located: a link connecting two nodes represents coupling between the two oscillators at those two nodes. Now, let us consider a SW networks where each node i ($i = 1, 2, \dots, N$) is an oscillator characterized by an angular phase x_i, and an intrinsic frequency w_i. The individual dynamics of the ith node is given by the differential equation:

$$\dot{x}_i(t) = w_i + K \sum_{j \in \Omega_i} \sin(x_j(t) - x_i(t)) \quad (i = 1, 2, \dots\dots, N) \tag{1}$$

where Ω_i is the set of neighbors of the node i and K is the coupling strength. In this paper, the intrinsic frequency w_i and the initial values of x_i are randomly drawn from the uniform distribution in the interval $(-1/2, 1/2)$ and $(-\pi, \pi)$, respectively. In original Kuramoto model which the system is globally coupled, the onset of synchronization emerges when the coupling strength approaches $K_c = 2/\pi g(w_0)$, where $g(w)$ is the distribution of the intrinsic frequency. Collective behavior of the coupled oscillators on SW networks is conveniently characterized by the following order parameter:

$$r(t) = \left[\left\langle \left| \frac{1}{N} \sum_{j=1}^{N} e^{ix_j(t)} \right| \right\rangle \right] \tag{2}$$

where $\langle \dots \rangle$ and $[\dots]$ represent the averages over time and over different realizations of the intrinsic frequencies, respectively, and $0 \leqslant r(t) \leqslant 1$. We discuss the synchronization by probability measure in this paper. That is, when $t \to \infty$, the system is said to be synchronized in probability if for any $x(t)$ and $\varepsilon > 0$, $P\{|1 - r(t)| \geq \varepsilon\} = 0$ exists.

3. Synchronization on SW networks

In the time-dependent Monte Carlo simulations performed in this work, we use $X = x(t)/\pi$ to measure the standard angular phase. Eq. (1) is solved by fourth-order Runge–Kutta method and the results are the average value of 30 times numerical experiments. The networks contain $N = 1000$ nodes, and average degree $\langle k \rangle = 10$. Qualitatively similar results are obtained with $N = 600$, $N = 800$ and $N = 2000$.