Physica A 416 (2014) 80–89

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Local rewiring rules for evolving complex networks

E.R. Colman [*], G.J. Rodgers

Department of Mathematical Sciences, Brunel University, Uxbridge, Middlesex UB8 3PH, UK

HIGHLIGHTS

- An analysis of the class of directed network with fixed out-degree.
- A stochastic model is introduced that incorporates growth, local and global rewiring.
- Degree distribution converges to an exponential form.
- When the rewiring is mostly local, small sets of nodes dominate and accumulate many links.

ARTICLE INFO

Article history:
Received 13 March 2014
Received in revised form 14 August 2014
Available online 26 August 2014

Keywords:
Random networks
Rewiring

ABSTRACT

The effects of link rewiring are considered for the class of directed networks where each node has the same fixed out-degree. We model a network generated by three mechanisms that are present in various networked systems; growth, global rewiring and local rewiring. During a rewiring phase a node is randomly selected, one of its out-going edges is detached from its destination then re-attached to the network in one of two possible ways; either globally to a randomly selected node, or locally to a descendant of a descendant of the originally selected node. Although the probability of attachment to a node increases with its connectivity, the probability of detachment also increases, the result is an exponential degree distribution with a small number of outlying nodes that have extremely large degree. We explain these outliers by identifying the circumstances for which a set of nodes can grow to very high degree.

1. Introduction

The question of how complex patterns can be produced by the collective behaviour of many interacting agents such as particles, cells or people, is one of the most important considerations in complexity science. The techniques of statistical physics that originated from the study of gasses and magnets have been adapted to address this question to explain a much wider range of emergent phenomena seen in biological and social systems. Fundamentally, mathematical models are used to derive statistical information about the system as a whole from the assumptions made about its constituent agents, or more specifically, the "rules" that govern their interactions. While in most physical systems agents interact with their closest neighbours in a spatial sense, many other systems are not constrained in this way, these are typically modelled as networks where the concept of distance between two points is redefined as the path-length between two nodes. An example of a local rule is triadic closure, the creation of a link between two nodes separated by a path-length of 2.

When the growth and evolution of a network is driven by local rules, nodes tend to be selected with a frequency proportional to how well connected they are. This is simply because a node with x connections is present in the neighbourhood

---

[*] Corresponding author. Tel.: +44 7821353915.
E-mail address: Ewan.Colman@brunel.ac.uk (E.R. Colman).

http://dx.doi.org/10.1016/j.physa.2014.08.046
0378-4371/© 2014 Published by Elsevier B.V.

of x other nodes, in other words there are x possible ways to discover the node via a local search. It is not surprising then, that the scale-free networks generated by global preferential attachment can also be created by numerous processes that use only local rules i.e. with no global knowledge of the network structure [1].

Typically in these models, a network will begin as a small set of nodes connected by edges, then with each iteration, more nodes are introduced and connections made, thus increasing the degree of those that are already there. Networks of this type are partly static in the sense that once an edge has been placed between two nodes it remains in that position for the rest of the network's lifetime. The class of network whose edges are dynamic, i.e. at any point could potentially be removed or rewired, has far wider scope of application.

This paper studies networks that combine dynamic edges with locally driven processes. Our model is an iterative process that evolves a network, the parameters are the rate of growth, and the rates of local and global (random) rewiring. We examine only networks with directed edges and nodes of a fixed out-going degree. For particular regions of the parameter space, we examine in detail a phenomenon whereby a small set of nodes, owing to their position in the network, gather significantly larger number of connections than those outside the set. These considerations lead to a good approximation of the extreme tail of the degree distribution, giving probabilities for the existence of outlying nodes of the distribution, sometimes referred to as dragon kings [2].

In Section 3 we introduce a model of growth and rewiring in directed networks and show the main results. The following sections describe the mathematical models and their solutions. In Section 4 we find the distribution of cycles of size n in the initial randomly wired graph. In Section 5 we find a formula for the degree distribution in the large t limit. In Section 6 we model the total degree of the dominant nodes and for selected parameter values derive the degree distribution tail.

2. Related work

Local rules for growing networks have been in the literature for some time [1,3]. In the model most similar to the one presented here [4], the preferential attachment mechanism is generalised to include rewiring events. They find both exponential and power-law degree distributions depending on the choice of parameters. Preferential attachment in rewiring has been studied on a network of fixed size with the interesting conclusion that a power law degree distribution can be achieved without a growing network [5]. This result relies on the use of a non-linear attachment kernel (heavily biased towards nodes with large degree) to ensure that nodes with large degree continue to grow in spite of the preferential detachment that also occurs through rewiring. This work has been extended to bipartite networks [6] which have an advantage of being free of degree correlations between neighbouring nodes, thus the results in Ref. [7] for the mean field solution to the degree distribution are exact. The same model also exhibits a condensation phenomenon, also know as gelation [3], where one node becomes connected to almost every other, this is relevant to the study of the dominant nodes presented here.

A large body of literature, much of which is commercially motivated, comes from the analysis of the network properties of web 2.0 systems [8,9]. We believe our results here are relevant in this field since rewiring, local dynamics and directed links are present in many of these self-organising systems. Twitter, for example, gives its users the option to "unfollow" other users meaning the edges are not static as they are in the majority of complex network models. Local rules, specifically triadic closure contribute to the growth of the network [10], however the distribution does not follow a power-law [11].

Recommendation algorithms designed to facilitate sharing online news articles, music, films, etc. connect users together based on the similarity of the content they have responded to positively. The content a user is exposed to in this way is limited to a small number of items shared by her neighbours. When the algorithm updates the links based on the most recent data, we can expect the strength of the similarity between her and her second neighbours to increase, making triadic closure likely. The network topologies of these networks have been studied in Ref. [12]. In this work the network is treated as a static object at one instant in time, clustering is found to be significantly higher than the random network which suggests that triadic closure could be part of the networks dynamics. The evolution of a theoretical model network [13] considers directed edges between "leaders" and "followers" that are rewired periodically according to a similarity score. A scale-free structure is found but the authors do not go into detail about the rewiring dynamics. The network evolution of recommendation networks perhaps deserves more attention since it exhibits cumulative advantage effects that have consequences for many commercial areas.

Our decision to restrict the model only to the case where every node has the same number of out-going links was motivated mostly by the considerable simplicity this would bring to the analysis. There is, however, some justification for this assumption regarding the suggested applications. Some product websites link each product to a fixed number of recommended products (amazon.com would be the most famous example although technically the number of recommended products is not fixed as it varies according to the size of the web browser). In the case of Twitter, it is sensible to assume that the number of accounts that a user will follow will, after enough time has passed, remain close to a steady value and not increase to infinity. Each user will differ in the number of other accounts they follow, but if we treat every user as an identical agent with the mean number of followings, then the model we present is appropriate.

3. Model and results

Let $G(N, mN)$ be a random graph in which each of the N nodes has m out-going directed edges, the destination of each directed edges is selected randomly. Throughout this paper we use 'degree' to refer to the in-coming degree of a node. In