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A percolation system with extremely long range connections and node dilution

M.L. de Almeida [a], E.L. Albuquerque [a], U.L. Fulco [a,*], M. Serva [a,b]

[a] Departamento de Biofísica e Farmacologia, Universidade Federal do Rio Grande do Norte, 59072-970, Natal-RN, Brazil

[b] Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università dell'Aquila, L'Aquila, Italy

HIGHLIGHTS

- We consider the bond-diluted long-range percolation problem on a linear chain.
- Dilution of nodes, which is also considered, competes with long-range connectivity.
- The percolation order parameter only depends on the average connectivity.
- The average connectivity is explicitly computed in terms of the free parameters.

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ABSTRACT

We study the very long-range bond-percolation problem on a linear chain with both node and bond dilution. Very long-range means that the probability p_{ij} for a connection between two nodes i, j at a distance r_{ij} decays as a power-law, i.e. $p_{ij} = \rho/[r_{ij}^{\alpha} N^{1-\alpha}]$ when $\alpha \in [0, 1)$, and $p_{ij} = \rho/[r_{ij} \ln(N)]$ when $\alpha = 1$. Node dilution means that the probability that a node is present in a site is $p_s \in (0, 1]$. The behavior of this model results from the competition between long-range connectivity which enhances the percolation, and node dilution which weakens percolation. The case $\alpha = 0$ with $p_s = 1$ is well-known, being the exactly solvable mean-field model. The percolation order parameter P_∞ is investigated numerically for different values of α, p_s and ρ. We show that in all range $\alpha \in [0, 1]$ the percolation order parameter P_∞ depends only on the average connectivity γ of the nodes, which can be explicitly computed in terms of the three parameters α, p_s and ρ.

1. Introduction

During the last fifty years, percolation theory has brought new understanding and methods to a broad range of topics in physics like materials science, complex networks, surface roughening, epidemiology, geography, and fire propagation (see Refs. [1,2] for a review). This theory was first considered for the optimization of masks supplied to the miners in the coal pits needing a protection which could block poisoning materials, while permitting the passage of air. In other words, it was needed an appropriate dosage of porosity of the material which composed the masks in order to have connected path for air and unconnected path for poisoning materials. After that, the theory was applied to the study of movement and filtering of fluids through porous materials (the most familiar phenomena probably being coffee percolation) and its scope has been progressively extended to all other domains [3–5].

* Corresponding author. Tel.: +55 84 3215 3419.
E-mail address: umbertofulco@gmail.com (U.L. Fulco).

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Nowadays, percolation is still a very active field of research in physics and applied to an always increasing number of phenomena as, for example, fluid flow in random media [6], dielectric breakdown [7] and reaction–diffusion processes in two-dimensional percolating structures [8].

Percolation models have also been increasingly adopted for many phenomena besides physics to understand important features of chemical, biological and social systems. Many of them form complex networks, whose vertices are the elements of the system and whose edges represent their interactions. For example, living systems form a huge genetic network whose vertices are proteins, while the edges represent their chemical exchanges [9]. Equally complex networks occur also in social science, where the vertices are individuals, organizations or countries and the edges characterize their social contacts [10]. Moreover, the effects of the complex connectivity of biological systems can be also studied by percolation theory. Recent advances in this field points to universal laws and offer a new conceptual framework that could potentially revolutionize our view of biology [11].

The effect of long-range connections on percolation is of fundamental interest, since they give rise to a variety of new interesting dynamical and thermodynamical phenomena. In view of that, long-range models have been intensively studied in recent times in different contexts [12–18]. The phenomenology becomes very interesting when long-range connections appear together with node dilution. In this case, in fact, there is competition between long-range connectivity which enhances percolation and node dilution which weaken it [19–21].

In this work we investigate the very long-range percolation problem on a linear chain with both node and bond dilution. Very long-range means that the probability p_{ij} of a connection between two nodes i, j at a distance r_{ij} decays as a power-law, i.e. $p_{ij} = \rho/[r_{ij}^{\alpha} N^{1-\alpha}]$ when $\alpha \in [0, 1)$ and $p_{ij} = \rho/[r_{ij} \ln(N)]$ when $\alpha = 1$. Node dilution means that the probability that a node is present in a site is $p_s \in (0, 1]$. Notice that for this very long-range models, in order to obtain the correct thermodynamic limit, it is compulsory to assume that the probability of a connection decays with the size N of the system as $1/N^{1-\alpha}$ (or as $1/\ln(N)$ in case $\alpha = 1$).

The case $\alpha = 0$, with $p_s = 1$, is well-known, being the exactly solvable mean-field model, while the case $\alpha = 0$ with $p_s < 1$ is its almost trivial extension. In the other regions, the percolation order parameter P_∞ is investigated numerically for different values of α, p_s and ρ. Intuitively, one expects the percolation order parameter P_∞ be reduced by the dilution of nodes [19,20]. Indeed, we will show not only that this is true, but we also show that in all range $\alpha \in [0, 1]$, the percolation order parameter P_∞ depends only on the average connectivity γ of nodes, which we explicitly compute in terms of the three parameters α, p_s and ρ.

In other words, given $\gamma = \gamma(\alpha, \rho, p_s)$, $P_\infty(\gamma)$ is always the same function, independently on the values of α, ρ and p_s. We stress that this result is not only true at the transition, but for all possible values of γ. Therefore, not only we state that the model is the universality class of mean-field bond-percolation (it would be an almost trivial result being it well known when node dilution is absent) but we prove that spatial structures are irrelevant for all values of parameters, being the average connectivity the only relevant aspect.

The paper is organized as follows: In Section 2 we consider the simple case $\alpha = 0$ in the absence and in the presence of dilution. Sections 3 and 4 discuss the cases $\alpha \in (0, 1)$ and $\alpha = 1$ respectively. Finally, our conclusions are in Section 5.

2. Mean-field ($\alpha = 0$)

The percolation order parameter P_∞ is defined as the fraction of nodes of the system that belongs to the infinite cluster. Obviously, P_∞ attains its maximum value ($P_\infty = 1$) when all the nodes are in the infinite cluster, whereas $P_\infty = 0$ below a certain threshold, when the infinite cluster is absent.

A particularly simple model is the mean-field, which corresponds to $\alpha = 0$. We describe below this almost trivial case, first when only bonds are diluted, and afterwards considering dilution for both nodes and bonds.

2.1. Mean-field (bond diluted)

In mean-field bond diluted model ($\alpha = 0,\ p_s = 1$), one assumes that there are N nodes. Any pair of nodes is connected (closed bond) with probability ρ/N and unconnected (open bond) with probability $1 - \rho/N$.

The average connectivity γ of a given node (the average number of connections of a node to the remaining $N - 1$ nodes) is given by

$$\gamma = \frac{\rho}{N}(N-1) \simeq \rho. \tag{1}$$

This number is simply obtained by multiplying the number $N-1$ of remaining nodes by the probability that a bond is closed.

Let us call P_∞ the fraction of nodes in the giant component (number of nodes in the giant component divided by the total number of nodes N), which can also be seen as the probability that a node belongs to the giant component itself. The order parameter P_∞ satisfies the self-consistency equation (see, for example, Ref. [22,23])

$$\exp(-\gamma P_\infty) = 1 - P_\infty, \tag{2}$$

whose solution $P_\infty(\gamma)$ is depicted in Fig. 1. The critical value of the control parameter is $\gamma_c = 1$.