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Thermodynamics of a morphological transition in a relativistic gas

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HIGHLIGHTS

- A morphological transition in the distribution function of a relativistic gas marks the transition from classical to relativistic behavior.
- We reformulate this transition in the language of thermodynamic theory of phase transition.
- We calculate critical exponents and the critical temperature.
- Relativistic constraint leads to emergence of order parameter as the most probable velocity becomes non-zero in the relativistic regime.
- The case of quantum statistics is also considered with their possible implications for real physical systems.

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ABSTRACT

Recently, a morphological transition in the velocity distribution of a relativistic gas has been pointed out which shows hallmarks of a critical phenomenon. Here, we provide a general framework which allows for a thermodynamic approach to such a critical phenomenon. We therefore construct a thermodynamic potential which upon expansion leads to Landau-like (mean-field) theory of phase transition. We are therefore able to calculate critical exponents and explain the spontaneous emergence of “order parameter” as a result of relativistic constraints. Numerical solutions which confirm our thermodynamic approach are also provided. Our approach provides a general understanding of such a transition as well as leading to some new results. Finally, we briefly discuss some possible physical consequences of our results as well as considering the case of quantum relativistic gases.

1. Introduction

As early as 1911, F. Jüttner provided a relativistic generalization of the famous Maxwell–Boltzmann (MB) velocity distribution whose Gaussian form does not respect the maximal velocity of light, c. He used entropy maximization principle under relativistic energy–momentum conservation constraints to arrive at what is now known as the Jüttner distribution [1]. For small enough temperatures where the typical velocities are small, the Jüttner distribution reduces to the classical MB distribution while for high temperatures the relativistic constraints forces it to take on a very different form. Although the validity of the Jüttner distribution has been a source of some controversies [2–7], it has been recently established as the correct relativistic velocity distribution [8–14]. However, one important question still remains: what is the temperature scale that determines whether a gas is relativistic or classical? A simple answer may be $\theta \equiv mc^2/k_BT \approx 1$ where clearly $\theta \gg 1$ is the classical and $\theta \ll 1$ the ultra-relativistic limits. Recently, it has been suggested that the Jüttner distribution exhibits a morphological transition at $\theta_c = d + 2$, where d is the spatial dimension of the relativistic gas [15]. That is, for

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$\theta > \theta_c$ the distribution function has a classical form while for $\theta \leq \theta_c$ it starts to exhibit an increasingly different form from the classical limit. While this transition temperature is much lower than naive expectation, the more interesting result was that such a transition exhibits some similarity to thermodynamic phase transitions with corresponding (d-independent) critical exponents [15]. However, no reasoning was given as to the origin of such a critical phenomenon. For example, why are the exponents d-independent? What is the physical significance of the order parameter? Is there a singular behavior (a true hallmark of criticality) as in a "generalized susceptibility?" Is there a symmetry breaking principle which causes such a phase transition? In the present work, we provide a general framework which connects the Jüttner distribution with the thermodynamic theory of phase transitions as described by the Landau theory. In doing so, not only we provide a general framework, we also provide simple answers to the above questions. We also note that such a transition and its physical properties may have important consequences in real world systems for which $\theta \lesssim 1$, which is not necessarily a very high temperature, for example in graphene as clearly explained in Ref. [15]. However, relativistic astrophysics [16,17] and high energy physics (e.g. quark–gluon plasma [18,19]) are also two important areas of active research where such results may have important consequences. In the following we use natural units, $c = k_B = 1$, and set $m = 1$ without loss of generality.

2. Results

Our starting point is a simple observation that in statistical mechanics, equilibrium distributions are related to the thermodynamic potentials via the relation $f = e^{-F}$, thus giving $F = -\ln f$, e.g. $f = \frac{1}{\Omega}$ in the entropy representation where all Ω micro-states are equally likely [20]. We therefore simply build such a generalized thermodynamic function using the Jüttner distribution and use it to calculate various thermodynamic relations. The Jüttner distribution is given by Refs. [9,10]:

$$f(\vec{v}, \vec{u}, T) = \frac{A\gamma^{d+2}(v)}{\gamma(u)\left[\exp\left(\frac{1-\vec{u}\cdot\vec{v}}{T}\gamma(u)\gamma(v) - \frac{\mu}{T}\right) + \lambda\right]}, \tag{1}$$

where $\gamma(v) = (1-v^2)^{-\frac{1}{2}}$ is the Lorentz factor and $u = |\vec{u}|$ is the average velocity, i.e., $\vec{u} = \langle\vec{v}\rangle$ which is taken to be zero in the co-moving frame, and A is a normalization constant. μ is the chemical potential and $\lambda = +1, -1, 0$ distinguishes the Fermi, Bose and Boltzmann statistics. For simplicity we set $\mu = \lambda = 0$ here. This leads to:

$$F(\vec{v}, \vec{u}, T) = -\ln f = -\ln\left[A\gamma^{-1}(u)\right] - (d+2)\ln\gamma(v) + \left[\frac{1-\vec{u}\cdot\vec{v}}{T}\gamma(v)\gamma(u)\right]. \tag{2}$$

Noting that $u, v \leq 1$ and that typically $u \ll 1$, we Taylor expand the above expression thus obtaining,

$$F(\vec{v}, \vec{u}, T) = \left[-\ln A + \frac{1}{T}\right] + \frac{1}{2}\left[\frac{1}{T} - (d+2)\right]v^2 + \frac{3}{8}\left[\frac{1}{T} - \frac{2}{3}(d+2)\right]v^4 - \frac{\vec{u}\cdot\vec{v}}{T} + \cdots. \tag{3}$$

Clearly, this has the same form as the Landau functional:

$$G(\phi, h, T) = g(h, T) + a(T)\phi^2 + b(T)\phi^4 - h\phi, \tag{4}$$

which describes the mean-field theory of critical phase transition at a temperature given by $a(T_c) = 0$, in the absence of "conjugate field", h. This immediately gives the result $T_c = 1/(d+2)$, consistent with the previous study [15]. Fig. 1 shows the Jüttner function and the corresponding thermodynamic function $F = -\ln f$ for various temperatures for $d = 1$. Note that the morphological transition corresponds to the appearance of new stable minima in the thermodynamic function. Accordingly, thermodynamic properties are obtained by entropy principle which extremizes the thermodynamic potential, leading to

$$\delta F = 0 = \left(\frac{1}{T} - \frac{1}{T_c}\right)v_{mp} + \left(\frac{3}{2T} - \frac{1}{T_c}\right)v_{mp}^3 - \frac{u}{T} + \cdots \tag{5}$$

which (for $u = 0$) gives:

$$v_{mp} = \begin{cases} 0; & t < 0 \\ \pm\sqrt{2t}; & \frac{1}{2} > t > 0 \end{cases} \tag{6}$$

where v_{mp} is the most probable velocity and $t \equiv (T - T_c)/T_c$, and the upper limit on t is due to the constraint that $v \leq 1$.

Comparing the above with the general Landau theory of phase transition, one immediately realizes that v_{mp} is the order parameter associated with a continuous (second-order) phase transition which occurs at $T = T_c = 1/(d+2)$ in the absence of the conjugate field u. Several thermodynamic relations follow immediately [21]:

$$v_{mp} \sim (T - T_c)^{\frac{1}{2}}; \quad (T \gtrsim T_c) \tag{7}$$

$$v_{mp} \sim u^{\frac{1}{3}}; \quad (T = T_c) \tag{8}$$

$$\chi \sim \lim_{u\to 0}\left(\frac{\partial v_{mp}}{\partial u}\right)_T \sim t^{-1}; \quad (|t| \ll 1) \tag{9}$$