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Hamilton–Jacobi and Fokker–Planck equations for the harmonic oscillator

Q1 Emilio Cortés *, J.I. Jiménez-Aquino

Departamento de Física, Universidad Autónoma Metropolitana—Iztapalapa, Apartado Postal 55-534, C.P. 09340, México, Distrito Federal, Mexico

HIGHLIGHTS

- Hamilton–Jacobi and Fokker–Planck equations are equivalent for harmonic oscillators.
- Extremal action supplies the solution of both equations.
- The problem in presence of a magnetic field is explicitly solved.

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ABSTRACT

Using Feynman's path integral formalism applied to stochastic classical processes, we show the equivalence between the Hamilton–Jacobi (HJ) and Fokker–Planck (FP) equations, associated with an overdamped Brownian harmonic oscillator. In this case, the Langevin equation leads to a Gaussian Lagrangian function and then the path integration which defines the conditional probability density can be replaced by the extremal path. Due to this fact and following the classical dynamics formalism, we prove the strict equivalence between the HJ and FP equations. We do this first for an ordinary Brownian harmonic oscillator and then the proof is extended to an electrically charged Brownian particle under the action of force fields: magnetic field and additional time-dependent force fields. We observe that this extremal action principle allows us to derive in a straightforward way not only the HJ differential equation, but also its solution, the extremal action.

1. Introduction

Feynman's path integral formalism (FPIF) [1] has been a useful tool to describe a variety of systems in the nonequilibrium statistical mechanics. It continues to be a very useful method in the study of classical [2–5] and quantum Brownian motion [6,7]. Between the eighties and nineties, the formalism was used to characterize the noise-induced linear and nonlinear stochastic dynamics driven by Gaussian colored noises [8]. The path integral formalism and its connection with the Hamiltonian dynamics has also been explored and applied to other situations where the stochastic fluctuations play a fundamental role [9]. In particular, in Ref. [10] a Hamiltonian formalism was given for a second order Langevin equation where an extremal action is formally written in terms of the nonlinear Hamilton equations. Due to the nonlinearity the expression for the extremal action in general is not easy to evaluate explicitly. Other works that use the Hamiltonian dynamics focus mainly on the probability distributions in the steady state, which are the time-independent solutions of the FP equations. It is shown that in the weak-noise limit of the steady state distribution, the FP equation reduces to a Hamilton–Jacobi-like equation [9,11].

* Corresponding author.
E-mail addresses: emil@xanum.uam.mx (E. Cortés), ines@xanum.uam.mx (J.I. Jiménez-Aquino).

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From quantum mechanics point of view, it would be worth to comment that the FPIF was also used to obtain explicitly the Schrödinger equation from the Hamilton–Jacobi equation [12]. However, to the best of knowledge the explicit equivalence between the HJ and FP equations has not been reported in the literature even for linear equations. Our main contribution in this work is to use also the FPIF to show the equivalence between the HJ and FP equations for an overdamped Brownian harmonic oscillator, which satisfies a Langevin equation with additive Gaussian white noise (thermal noise). The equivalence is first proven for an ordinary Brownian harmonic oscillator (OBHO) and then when the oscillator is electrically charged, under the action of force fields: magnetic field and time-dependent force fields [13–17]. The Lagrangian is Gaussian in both cases which allows us (as a necessary condition) to replace the sum over paths by the extremal path [1]. With the extremal action defined in a formal way we follow the Hamiltonian formalism to obtain both, the extremal action by solving the Hamilton equations and also the HJ equation as a partial differential equation for that function. Once the HJ equation has been obtained in both cases, we show that it is totally equivalent to the FP equation through the relation $W(x, t|x_0) = N(t)\exp(-S[\bar{x}]/4\lambda)$, where $W(x, t|x_0)$ is the transition probability density (TPD) associated with the FP equation, $S[\bar{x}]$ is the extremal action associated with the extremal path $\bar{x}$, and $N(t)$ is the normalization factor [18,19]. It must be noticed that the extremal action S is calculated for all time $t \geq 0$ and as a consequence the probability density W is also given for all time $t \geq 0$. Therefore, the equivalence means that the HJ equation is obtained from the FP equation not in the asymptotic regime but for all time $t \geq 0$; besides the former equation does not mean a small-noise limit of the latter. The explicit expression for the extremal action gives the immediate solution of the FP equation. As a matter of fact, this approach which yields the extremal action from the Hamilton equations, constitutes an alternative way to obtain the solution of the FP equation.

This work is then outlined as follows: in Section 2, we study the formalism of the extremal action and its corresponding HJ equation for an overdamped OBHO in the one dimensional case. The necessary and sufficient condition for the equivalence between the HJ and FP equations is established. Once the extremal action is calculated, the well known solution of the FP equation is easily verified. In Section 3, we extend the formalism of Section 2 to the case of an overdamped charged Brownian harmonic oscillator (CBHO) in the presence of a constant magnetic field only. In this case, the equivalence between the HJ and FP equations is established by a similar condition as in the OBHO. Also the solution of the FP equation is immediately obtained. The problem of an overdamped CBHO in the presence of additional time-dependent force fields is studied in Section 4. Our concluding remarks are given in Section 5, and at the end of our work two Appendices are included for explicit calculations.

2. Extremal action and HJ equation for an OBHO

We start from a stochastic differential equation (SDE) given by a Langevin equation for a free harmonic particle in one dimension in the non-inertial regime (overdamped approximation)

$$\gamma\dot{x} + \omega^2 x = \xi(t), \tag{1}$$

where $\gamma = \alpha/m$ is the friction coefficient α per unit mass, $\omega^2 = k/m$ the oscillator's characteristic frequency, and $\xi(t)$ the noise per unit mass. Let us assume a Gaussian noise with zero first moment and delta correlation function

$$\langle\xi(t)\xi(t')\rangle = 2\lambda\,\delta(t - t'), \tag{2}$$

where λ is the noise intensity which satisfies the fluctuation–dissipation relation $\lambda = \gamma k_B T/m$, with k_B the Boltzmann constant and T the temperature of the surrounding medium (the thermal bath). In order to calculate the extremal action of this problem, we follow Feynman's functional formalism [1] for a stochastic process. Due to the fact that this stochastic differential equation is linear with additive noise, it turns out that noise probability distribution has a one-to-one correspondence with the distribution of the dynamical variable $x(t)$. Taking into account the Gaussian character of the noise, then we start from a Lagrangian function which represents the systematic part of the Langevin equation.[1]

$$L(\dot{x}(t), x(t)) = (\gamma\dot{x} + \omega^2 x)^2, \tag{3}$$

and the action functional reads

$$S[x(t)] = \int_{t_1}^{t_2} L(\dot{x}(t), x(t))\,dt. \tag{4}$$

The integral is evaluated along a path $x(t)$ with fixed end points (x_1, t_1) and (x_2, t_2). From this, the conditional probability density can be formally defined as a path integral

$$W(x, t|x_0) = N(t)\int \exp([-S[x(t)]/4\lambda])\,\mathcal{D}[x(t)], \tag{5}$$

Q2 in which $\mathcal{D}[x(t)]$ means a path differential and $N(t)$ is a normalization factor. According to Feynman and Hibbs [1], if $L(\dot{x}(t), x(t))$ is Gaussian, which means a function up to second degree in its variables, then the sum over the paths can

[1] Here the Lagrangian function has a different definition from that of classical dynamics.