Physica A 411 (2014) 128–137

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Trichotomous noise controlled signal amplification in a generalized Verhulst model

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HIGHLIGHTS

- Interplay of the seasonal and stochastic variabilities in population growth models is discussed.
- The approach is based on the theory of stochastic processes.
- Expressions for the probability distribution and the moments of the population size are presented.
- Noise parameters controlled amplification of a small input signal is demonstrated.

ARTICLE INFO

Article history:
Received 25 March 2014
Received in revised form 11 June 2014
Available online 20 June 2014

Keywords:
Noise-generated amplification
Verhulst self-regulation
Colored noise
Symbiotic interaction

ABSTRACT

The long-time limit of the probability distribution and statistical moments for a population size are studied by means of a stochastic growth model with generalized Verhulst self-regulation. The effect of variable environment on the carrying capacity of a population is modeled by a multiplicative three-level Markovian noise and by a time periodic deterministic component. Exact expressions for the moments of the population size have been calculated. It is shown that an interplay of a small periodic forcing and colored noise can cause large oscillations of the mean population size. The conditions for the appearance of such a phenomenon are found and illustrated by graphs. Implications of the results on models of symbiotic metapopulations are also discussed. Particularly, it is demonstrated that the effect of noise-generated amplification of an input signal gets more pronounced as the intensity of symbiotic interaction increases.

1. Introduction

A wide range of natural processes occurs under the influence of external time-periodic and random forcing [1]. In recent years, an increasing interest has been shown in non-equilibrium noise-driven phenomena, of which stochastic resonance [2–5], noise-enhanced stability [6–8], stochastic transport in ratchets [9–11], and memory-induced resonance in fractional oscillators [12,13] are a few examples. Active analytical and numerical investigations of various models in this field have been stimulated by their possible applications in population biology, molecular biology, chemical physics, nanotechnology, and for separation techniques of nanoobjects [9,14].

One of the objects of special attention in this context is the noise-driven population growth problem. It is crucial not only to be able to describe ecological systems, like bacterial or animal populations, but also to understand the behavior of other systems, that can be described by non-negative integer counting, with members of different species acting within the problem scope. Molecules, atoms, photons, particles of plasma, predators and prey, infected individuals, etc. can all be regarded as populations under a diversity of situations [15,16].

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http://dx.doi.org/10.1016/j.physa.2014.06.016
0378-4371/

Deterministic models of population growth dynamics based on a generalized Verhulst self-regulation mechanism (GVM) are the standard framework for a wide array of assemblies that consist of a number of elements that interact through cooperative or competitive mechanisms [17]. Several particular cases of GVM are of great use in different applications [18], such as the growth process of a biological population [17,19], autocatalytic chemical reactions [20], biochemical systems [21], laser physics [22], social sciences [23], etc. In spite of these achievements, deterministic models of population growth dynamics cannot directly handle a range of situations dominated by environmental fluctuations in the forcing or in the control parameters, which are intrinsic to the phenomenon [1]. For example, several environmental systems can be described by state variables representing the availability of a resource whose dynamics is forced by climatic oscillations and diverse stochastic environmental factors. Particularly, in GVM the carrying capacity (a limiting factor on the population growth, which is imposed by environmental factors, basically food and space limitations) may vary over time. The most productive abstraction of noise-like influence from the environment is the case of Gaussian white noise. However, time-series of real environmental variables contain positive temporal autocorrelation [24,25], which is an indication that actual environmental fluctuations are colored. Although the effect of environmental colored noise on the carrying capacity, and thus also on population dynamics have been subject to intense theoretical investigations [26,27], it seems that proper analysis of the potential consequences of an interplay of colored fluctuation and a time-periodic seasonal drive of the carrying capacity in population growth models with a GVM is still missing in the literature.

Thus motivated, we consider a stochastic population growth model subjected to a GVM. The influence of a variable environment on the carrying capacity is modeled by a deterministic time-periodic component and by a colored trichotomous noise. Although both dichotomous and trichotomous noises may be useful in modeling natural colored fluctuations, the latter is more flexible, covering all cases of dichotomous noise [28].

The main purpose of this paper is to provide, in the long-time limit, exact formulas for the analytical treatment of the behavior of the probability distribution and the statistical moments of the population size. Particularly, we show that the interplay of colored fluctuations and small deterministic oscillations of environmental parameters can significantly amplify the oscillations of the mean population size. Implications of this phenomenon on models of symbiotic metapopulations are also discussed.

To avoid misunderstanding, let us mention that this phenomenon is qualitatively different from the conventional stochastic resonance [2,4,29]. In the case of stochastic resonance, the amplification of a small input signal means a nonmonotonic dependence of the output signal or some function thereof on noise parameters, but in the case considered here the amplification depends monotonically on noise parameters.

2. Model

Our starting point is the generalized Verhulst equation

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = X(t)g(X(t)), \tag{1}$$

where $X(t) > 0$ is the population abundance and the function $g(X)$ describes the selfregulation of the population. Typical mechanisms for selfregulation are, for example, territorial breeding requirements and the crowding effect caused by the competition for resources [30]. These are taken into account by applying the GVM

$$g(X) = \delta - \gamma(t)X^{\beta}, \tag{2}$$

with $\beta > 0$, where $\gamma(t)$ characterizes the carrying capacity for the population and $\delta > 0$ is the growth rate parameter of the species [30,31]. The effect of time-variable environment on the dynamics of the population is taken into account as a variable carrying capacity in the model including a time-periodic deterministic part and a three-level Markovian noise (trichotomous noise). More precisely we assume that

$$\gamma(t) = \gamma\left(1 + A_0 \sin(\Omega t) + Z(t)\right), \tag{3}$$

where the constant γ is the mean value of $\gamma(t)$ which determines the carrying capacity K without any temporal perturbations:

$$K = \left(\frac{\delta}{\gamma}\right)^{\frac{1}{\beta}}. \tag{4}$$

The periodic term with an amplitude A_0 and the frequency Ω in the right hand side of Eq. (3) mimics the time dependence of the availability of a resource whose dynamics is forced by diverse environmental factors and climatic oscillations. Random interaction with the environment (climate, disease, etc.) is taken into account by introducing a colored noise $Z(t)$ in $\gamma(t)$.

The colored noise $Z(t)$ is assumed to be a zero-mean trichotomous process [28]. The trichotomous process is a random stationary Markovian process that consists of jumps between three values a, 0, and $-a$. The jumps follow in time according to a Poisson process, while the values occur with the stationary probabilities

$$p_s(a) = p_s(-a) = q, \qquad p_s(0) = 1 - 2q, \tag{5}$$

with $0 < q \le 1/2$. The mean value of $Z(t)$ and the correlation function are

$$\langle Z(t)\rangle = 0, \quad \langle Z(t+\tau)Z(t)\rangle = 2qa^2\mathrm{e}^{-\nu\tau}. \tag{6}$$