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First passage time distributions of anomalous biased diffusion with double absorbing barriers

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HIGHLIGHTS

- We investigate the first passage time problem of the Galilei variant sub-diffusion.
- Explicit solutions in the semi-infinite and finite domains are obtained.
- Asymptotic behavior of the first passage time is confirmed by numerical results.

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ABSTRACT

We investigate the first passage time (FPT) problem of anomalous diffusion governed by the Galilei variant fractional diffusion–advection equation in the semi-infinite and finite domains subject to an absorbing boundary condition. We obtain explicit solutions for the FPT distributions and the corresponding Laplace transforms for both zero and constant drift cases by using the method of separation of variables as well as the properties of the Fox H function. An important relation between the FPT distributions corresponding to one and two absorbing barriers is revealed to determine the conditional FPT distributions. It shows that the proportion between the conditional FPT distributions only depends on the general Péclet number. We further discuss the asymptotic behavior of the FPT distributions and confirm our theoretical analysis by numerical results.

1. Introduction

Diffusion is one of the most important phenomena encountered in numerous physical, chemical and biological systems [1]. However, the picture that has emerged over the last few decades clearly reveals that an increasing number of natural phenomena do not fit into the relatively simple description of normal diffusion [2]. Anomalous diffusion turns out to be quite ubiquitous and it is characterized by a nonlinear behavior for the mean square displacement in the course of time [3]. In general, there are several approaches, such as fractional diffusion equations and continuous time random walk models, to describe anomalous diffusion processes [4,5].

As far as the stochastic behavior of anomalous diffusion is concerned, the first passage time (FPT) problem plays an important role when the microscopic mechanisms or statistical properties are investigated [6–11]. Particularly, explicit FPT distributions are important for lattice Monte Carlo simulation of normal or anomalous diffusion [12–15]. The focus of this study is on the FPT distributions of anomalous diffusion governed by the Galilei variant fractional diffusion–advection

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equation (FDAE) model [5] subject to an absorbing boundary condition. This model can be derived from continuous time random walks with power-law waiting time distribution and diverging time scale [16]. In contrast to other models for anomalous diffusion, this model can be used to describe physical systems where trapping occurs, i.e., the particle gets repeatedly immobilized before it gets dragged along the velocity stream again, and anomalous diffusion processes in porous media [5].

For anomalous diffusion governed by the Galilei variant FDAE, the Laplace transform of the FPT distribution in the semi-infinite domain with absorbing boundary can be obtained for the case of a constant drift [17]. Conversely, in the absence of a drift, an explicit solution for the FPT distribution in the finite domain can be given in terms of different series [18,19]. Our objective is to obtain explicit solutions for the FPT distributions and the corresponding Laplace transforms in the semi-infinite and finite domains subject to an absorbing boundary condition for both cases with and without the constant drift. The rest of the paper is organized as follows. In Section 2, we follow the method of separation of variables [17,20] to obtain the FPT distribution with one absorbing barrier. The result will be used to derive the conditional FPT distribution with double absorbing barriers. In Section 3, we first obtain the Laplace transforms of the conditional and unconditional FPT distributions with double absorbing barriers. Then we proceed to give explicit solutions for the FPT distributions by using the properties of the Fox H function. In Section 4, we further discuss the asymptotic behavior of the FPT distributions and confirm the theoretical analysis by numerical results. Finally, we summarize the work and draw some conclusions.

2. First passage time distribution in the semi-infinite domain

For anomalous diffusion governed by the Galilei variant FDAE, the FPT problem can be recast as a boundary value problem subject to an absorbing boundary condition. Then it is not difficult to obtain the Laplace transform of the FPT distribution using the method of separation of variables. For the case of a constant drift, the Galilei variant FDAE is given by Ref. [5]

$$\frac{\partial}{\partial t}P(x,t) = {}_0\partial_t^{1-\alpha}\left(D_\alpha\frac{\partial^2}{\partial x^2}P(x,t) + \upsilon_\alpha\frac{\partial}{\partial x}P(x,t)\right), \tag{1}$$

with the following boundary and initial conditions:

$$P(0,t)=0, \qquad P(+\infty,t)=0, \qquad P(x,0)=\delta(x-\ell), \quad \ell>0, \tag{2}$$

and the Riemann–Liouville operator is defined as

$${}_0\partial_t^{1-\alpha}P(x,t) = \frac{\partial}{\partial t}{}_0\partial_t^{-\alpha}P(x,t) = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t (t-\tau)^{\alpha-1}P(x,\tau)\mathrm{d}\tau. \tag{3}$$

Note that the generalized diffusion coefficient D_α has the dimension $[D_\alpha] = (\text{length})^2\cdot(\text{time})^{-\alpha}$ and the generalized velocity υ_α has the dimension $[\upsilon_\alpha] = (\text{length})\cdot(\text{time})^{-\alpha}$. Let $P(x,t) = X(x)T(t)$. Substituting in Eq. (1) we obtain

$$X(x)\frac{\mathrm{d}}{\mathrm{d}t}T(t) = {}_0\partial_t^{1-\alpha}T(t)\left(D_\alpha\frac{\mathrm{d}^2}{\mathrm{d}x^2}X(x) + \upsilon_\alpha\frac{\mathrm{d}}{\mathrm{d}x}X(x)\right). \tag{4}$$

Separating out the variables and introducing the separation constant λ we have

$$D_\alpha\frac{\mathrm{d}^2}{\mathrm{d}x^2}X(x) + \upsilon_\alpha\frac{\mathrm{d}}{\mathrm{d}x}X(x) = \lambda X(x), \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}T(t) = \lambda_0\partial_t^{1-\alpha}T(t). \tag{6}$$

With the boundary and initial conditions given in Eq. (2), the solution of Eq. (1) is given by the following integral over λ:

$$P(x,t) = \frac{2}{\pi}\int_{\upsilon_\alpha^2/4D_\alpha}^{\infty}\exp\left(-\frac{\upsilon_\alpha(x-\ell)}{2D_\alpha}\right)\sin\left(\ell\sqrt{4\lambda D_\alpha-\upsilon_\alpha^2}/2D_\alpha\right)\frac{\sin\left(x\sqrt{4\lambda D_\alpha-\upsilon_\alpha^2}/2D_\alpha\right)}{\sqrt{4\lambda D_\alpha-\upsilon_\alpha^2}}E_\alpha(-\lambda t^\alpha)\mathrm{d}\lambda, \tag{7}$$

where $E_\alpha(z)$ is the usual Mittag-Leffler function defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(1+\alpha n)}. \tag{8}$$

Let $\zeta = \sqrt{4\lambda D_\alpha-\upsilon_\alpha^2}/2D_\alpha$. Eq. (7) can be rewritten as

$$\begin{aligned} P(x,t) &= \frac{2}{\pi}\int_0^\infty \exp\left(-\frac{\upsilon_\alpha(x-\ell)}{2D_\alpha}\right)\sin(\ell\zeta)\sin(x\zeta)\,E_\alpha\left(-\left(D_\alpha\zeta^2+\frac{\upsilon_\alpha^2}{4D_\alpha}\right)t^\alpha\right)\mathrm{d}\zeta \\ &= \frac{1}{\pi}\int_0^\infty \exp\left(-\frac{\upsilon_\alpha(x-\ell)}{2D_\alpha}\right)(\cos((x-\ell)\zeta)-\cos((x+\ell)\zeta))\,E_\alpha\left(-\left(D_\alpha\zeta^2+\frac{\upsilon_\alpha^2}{4D_\alpha}\right)t^\alpha\right)\mathrm{d}\zeta. \end{aligned} \tag{9}$$