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# On the dynamical vs. thermodynamical performance of a $\beta$-type Stirling engine

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## HIGHLIGHTS

- A mathematical model for a $\beta$-type Stirling engine is introduced.
- The model is simple but includes all relevant thermodynamic and mechanical aspects.
- We obtain sufficient conditions for engine cycling and model stability characteristics.
- The performance of the engine's thermodynamic part is also investigated.

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## ABSTRACT

In this work we present a simple mathematical model for a $\beta$-type Stirling engine. Despite its simplicity, the model considers all the engine's relevant thermodynamic and mechanical aspects. The dynamic behavior of the model equation of motion is analyzed in order to obtain the sufficient conditions for engine cycling and to study the stability of the stationary regime. The performance of the engine's thermodynamic part is also investigated. As a matter of fact, we found that it corresponds to a Carnot engine.



## 1. Introduction

Arguably, the book *Reflections on the Motive Power of Fire and on Machines Fitted to Develop that Power* (*Réflexions sur la puissance motrice du feu et sur les machines propres à développer cette puissance* in French in the original) by Nicolas Léonard Sadi Carnot [1] is the birth certificate of thermodynamics. In this book, Carnot introduced the celebrated cycle bearing his name. The so-called Carnot cycle has greatly influenced the development of thermodynamics. Not only it catalyzed the discovery of the first and second laws of thermodynamics, but it has also served as an archetype for all thermal engines. Other thermal cycles have been proposed to understand the performance of real thermal engines: Otto cycle, Diesel cycle, Stirling cycle, etc. All of them have in common that they pose bonds for the thermodynamic performance of the corresponding engines. One can say that thermodynamic cycles model the functioning of the thermodynamic component of the corresponding real engines under ideal conditions.

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Continuing with the discussion in the previous paragraph, it can be asserted that, when they are regarded as models for thermal engines, ideal thermal cycles lack two important features: they ignore all possible ways in which real engines deviate from ideal thermodynamic conditions, and they omit the engines' mechanical components. Numerous efforts have been made to develop models for thermal engines that incorporate non ideal thermodynamical aspects. Consider for instance all the work carried out in the field of finite-time thermodynamics [2,3]. On the other hand, although in the engineering literature there is no scarcity of models that consider in full detail different mechanical aspects of thermal engines, to the best of our knowledge there are not enough simple models that consider both their thermodynamics and mechanical features. Some examples of the above for the case of Otto engines are [4–6]. To our consideration these simple models are necessary for at least two reasons: (1) they could provide the physics or engineering students with a complete yet simple picture of real thermal engines, thus easing the understanding of the thermo-mechanical coupling subtleties, and (2) these simple models could provide a framework out of which more detailed and predictive models can be built by incorporating specific details.

The present work is aimed at fulfilling, at least in part, the previously discussed necessity. We decided to focus in one of the simplest thermal engines ever build: the $\beta$-type Stirling engine. Already in 1983 Chen and Griffin [7] stated that most of the existing models for Stirling engines were limited to kinematic engines with emphasis on thermodynamic analysis, and that dynamic analysis should be integrated into thermodynamic study in future modeling efforts. Despite current efforts [8,9] we believe that a model that takes into consideration all the relevant thermodynamic and their mechanical characteristics, yet it is simple enough to allow a thorough analysis, is still lacking. The paper is organized as follows. Section 2 is devoted to developing a dynamical mathematical model for a $\beta$-type Stirling engine that considers all relevant thermodynamical and mechanical elements. In Section 3, the dynamic behavior of the resulting equation of motion is analyzed in detail to find the sufficient conditions for engine cycling, as well as the stability of the stationary regime. The stability analysis is inspired on previous stability studies on finite-time thermodynamics models [10–13]. In Section 4 the performance of the thermodynamic part of the engine is studied. And, finally, the obtained results are discussed and the derived conclusions are presented in Section 5.

## 2. Model development

*Geometrical considerations*

Consider the model for the $\beta$-type Stirling engine schematically represented in Fig. 1 [14]. Notice that the cylinder total volume is variable and determined by the upper piston position (here denoted by $x$). The lower piston divides the cylinder in two compartments. Assume that the gas in the upper compartment is kept at temperature $T_C$ at all times, while the gas in the lower compartment is kept at temperature $T_H$ ($T_H > T_C$).

Both pistons are connected to a wheel or radius $R$ through vertical arms and moving levers. The connecting points of the levers have an angular separation of $\pi/2$, as indicated in Fig. 1. In order for the arms not to interfere with the wheel movement, the length of the levers has to be at least $2R$. Here we assume that it is exactly equal to $2R$.

Let $\theta$ be the angular position (with respect to the vertical-up direction) of the attachment point of the upper-piston lever, and let $\alpha$ be the angle between the lever itself and the vertical-up direction. It is not hard to demonstrate from trigonometric considerations that these angles satisfy the following relations:

$$\sin\alpha = \frac{\sin\theta}{2}, \qquad \cos\alpha = \sqrt{1 - \frac{\sin^2\theta}{4}}. \tag{1}$$

From this, the positions of the upper and lower pistons can be written as functions of $\theta$ (the angular position of the upper-piston attachment point) as follows,

$$x(\theta) = R\left(a + \cos\theta - \sqrt{4 - \sin^2\theta}\right), \tag{2}$$

$$\begin{aligned} y(\theta) &= R\left(b + \cos(\theta + \pi/2) - \sqrt{4 - \sin^2(\theta + \pi/2)}\right) \\ &= R\left(b - \sin\theta - \sqrt{4 - \cos^2\theta}\right), \end{aligned} \tag{3}$$

in which $a$ and $b$ are constants whose value is determined by the length of the arms respectively connected to the upper and lower pistons, and by the choice of the system reference point. If the reference point is such that $y = 0$ at its minimum value and we ask that $x = y$ at a single point, then the $a$ and $b$ values ought to be given by

$$a = 4.5022311105795421, \qquad b = 3.$$

In Fig. 2 we plot the resulting $x(\theta)$ and $y(\theta)$ functions vs. $\theta$.