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Fidelity susceptibility and quantum Fisher information for density operators with arbitrary ranks

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HIGHLIGHTS

- The fidelity susceptibility is proportional to the quantum Fisher information.
- The fidelity susceptibility can be determined by the density matrix's support.
- The quantum Fisher information matrix can be determined by the density matrix's support.

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ABSTRACT

Taking into account the density matrices with non-full ranks, we show that the fidelity susceptibility is determined by the support of the density matrix. Combining with the corresponding expression of the quantum Fisher information, we rigorously prove that the fidelity susceptibility is proportional to the quantum Fisher information. As this proof can be naturally extended to the full rank case, this proportional relation is generally established for density matrices with arbitrary ranks. Furthermore, we give an analytical expression of the quantum Fisher information matrix, and show that it can also be represented in the density matrix's support.

1. Introduction

Q2

Quantum Fisher information (QFI) is the central concept in quantum metrology [1–15]. It depicts the theoretical bound of the variance of an estimator [16,17]

$$\mathrm{Var}(\hat{\theta}) \geq \frac{1}{F}. \tag{1}$$

Here $\hat{\theta}$ is the estimator for the parameter θ, $\mathrm{Var}(\cdot)$ describes the variance, and F is the so-called quantum Fisher information. A related concept widely used in quantum physics is the fidelity, which was first introduced by Uhlmann in 1976 [18]. For a parametrized state $\rho(\theta)$ and its neighbor state in parameter space $\rho(\theta+\delta\theta)$, where $\delta\theta$ is a small change of θ, the fidelity is defined as

$$f(\theta, \theta+\delta\theta) := \mathrm{Tr}\sqrt{\sqrt{\rho(\theta)}\rho(\theta+\delta\theta)\sqrt{\rho(\theta)}}. \tag{2}$$

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The form of fidelity is not unique, several alternative forms of fidelity have been proposed and discussed [19–21]. However, the Uhlmann fidelity is the most well-used form because it has a natural relation with Bures distance D_b, which is, $D_b^2 = 2 - 2f$. The fidelity only refers to the Uhlmann fidelity in this paper. The fidelity in Eq. (2) reveals the distinguishability between state $\rho(\theta)$ and state $\rho(\theta+\delta\theta)$. It depends on the small change parameter $\delta\theta$. To avoid this dependence, the concept of fidelity susceptibility (FS) is introduced [22]. It is generally known that the first-order term of $\delta\theta$ in fidelity is zero [23,24], thus FS is determined by the second-order term with the definition

$$\chi_f := -\frac{\partial^2 f(\theta, \theta+\delta\theta)}{\partial(\delta\theta)^2}. \tag{3}$$

FS is a more effective tool than fidelity itself in quantum physics, especially in detecting the quantum phase transitions [22,25–34]. Because of the relation between the fidelity and Bures distance, the nature of the fidelity susceptibility is actually the infinitesimal Bures distance.

Interestingly, the fidelity susceptibility and the quantum Fisher information are in fact closely related to each other. For a given state, the expression of FS is proportional to that of QFI [23,24,35–38]. Recently, we have obtained the expression of the QFI for a non-full rank density matrix ρ, which is determined by ρ's support [39]. The density matrix with non-full rank does physically exist in practice. For example, the input state of a quantum interferometer is always generated through a quantum process, which may practically not be perfect that the quantum decoherence and dissipation would wash out the diagonal and even the non-diagonal elements of the initial density matrix, resulting in a density matrix with non-full rank [40]. So it is physically necessary to consider the density matrices with non-full ranks. The fact that the QFI is determined by the density matrix's support makes us wonder that if the FS can be written in a similar way. In this paper, we give a detailed calculation of the fidelity for a non-full rank density matrix. We find that FS is also determined by the support of the density matrix. The whole calculation is rigorous and the expression of FS is proportional to that of QFI. Our calculation can be easily extended to the full-rank case. In addition, inspired by this result, we further study the quantum Fisher information matrix (QFIM), which is the counterpart of the QFI for the multiple-parameter estimations. Through the calculation, we find that the QFIM is also determined by the support of the density matrix.

The paper is organized as follows. In Section 2, for a non-full rank density matrix, we give the detailed calculation of the fidelity and obtain the expression of the FS, which is determined by the density matrix's support, and proportional to the expression of QFI. In addition, we apply the expression of QFI (or FS) to a non-full rank X state. In Section 3, we give the calculation of the QFIM and show that like QFI, QFIM is also determined by the support of the density matrix. We also apply this expression to a multiple parametrized X state with non-full rank. Section 4 is the conclusion of this work.

2. Proportional relationship between FS and QFI

In the following, we derive the expression of fidelity for a non-full rank density matrix. From which, we find the first-order term of fidelity vanishes. Then we get the expression of the FS, which is determined by the support of the density matrix. With the corresponding expression of the QFI, we prove the proportional relationship between FS and QFI. Although our proof concentrates on the density matrices with non-full ranks, it could be extended to the ones with full ranks as well.

2.1. Proof the proportional relationship

We will first obtain the expression of FS from the definition of fidelity in Eq. (2). For brevity, we rewrite that expression as $f = \mathrm{Tr}\sqrt{\mathcal{M}}$ with $\mathcal{M} := \sqrt{\rho(\theta)}\rho(\theta+\delta\theta)\sqrt{\rho(\theta)}$. We start our calculation by expanding $\rho(\theta+\delta\theta)$ up to the second order of the small change $\delta\theta$ as $\rho(\theta+\delta\theta) = \rho(\theta) + \partial_\theta\rho\delta\theta + \frac{1}{2}\partial_\theta^2\rho\delta^2\theta$ with $\partial_\theta\rho := \partial\rho/\partial\theta$ and $\partial_\theta^2\rho := \partial^2\rho/\partial\theta^2$. Then the matrix $\mathcal{M}$ takes the form

$$\mathcal{M} = \rho^2(\theta) + \mathcal{A}\delta\theta + \frac{1}{2}\mathcal{B}\delta^2\theta \tag{4}$$

where $\mathcal{A} = \sqrt{\rho(\theta)}\partial_\theta\rho\sqrt{\rho(\theta)}$ and $\mathcal{B} = \sqrt{\rho(\theta)}\partial_\theta^2\rho\sqrt{\rho(\theta)}$. This allows us to assume the square root of $\mathcal{M}$ in the form like

$$\sqrt{\mathcal{M}} = \rho(\theta) + \mathcal{X}\delta\theta + \mathcal{Y}\delta^2\theta, \tag{5}$$

which is also up to the second-order term of $\delta\theta$. As a result, taking square of both sides of Eq. (5), one can find the relations

$$\mathcal{A} = \rho\mathcal{X} + \mathcal{X}\rho, \tag{6}$$

$$\frac{1}{2}\mathcal{B} = \rho\mathcal{Y} + \mathcal{Y}\rho + \mathcal{X}^2. \tag{7}$$

Once the matrices $\mathcal{A}$ and $\mathcal{B}$ are obtained, the information of the matrices $\mathcal{X}$ and $\mathcal{Y}$ will be extracted from these two relationships.

Consequently, the expression of fidelity could be achieved from Eq. (5) as

$$f = 1 + \mathrm{Tr}(\mathcal{X})\delta\theta + \mathrm{Tr}(\mathcal{Y})\delta^2\theta. \tag{8}$$

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