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Thermodynamic approach to vortex production and diffusion in inhomogeneous superfluid turbulence

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HIGHLIGHTS

- A non-equilibrium thermodynamic framework is used for inhomogeneous quantum turbulence.
- A non-local model is proposed to describe inhomogeneous vortex tangles.
- The model describes vortex diffusion and new contributions to vortex formation.
- The entropy and entropy flux are compared with those in the maximum-entropy formalism.
- An experiment is proposed to check the non-local terms.

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ABSTRACT

In this paper, we use a non-equilibrium thermodynamic framework to generalize a previous nonlocal model of counterflow superfluid turbulence to incorporate some new coupled terms which may be relevant in the evolution of inhomogeneous vortex tangles. The theory chooses as fundamental fields the energy density, the heat flux, and the averaged vortex line length per unit volume. The constitutive quantities are assumed to depend on the fundamental fields and on their first spatial derivatives, allowing us to describe thermal dissipation, vortex diffusion and a new contribution to vortex formation. The restrictions on the constitutive relations are deduced from the entropy principle, using the Liu method of Lagrange multipliers. The form of the entropy and the entropy flux are compared with those obtained in the maximum-entropy formalism. The several non-local terms are discussed in detail to clarify their physical meaning, and an experiment is proposed to check their actual existence and measure their values.

1. Introduction

It is known that the presence of a heat flow in superfluid helium II causes the formation of a tangle of quantized vortex lines, which form, move, collide, and rearrange inside the superfluid [1–5]. The quantum of circulation is κ, given by $\kappa = h/m_4$, with h the Planck constant, and m_4 the mass of ^{4}He atom. The tangle is usually assumed to be homogeneous and isotropic. This simplifies the mathematical analysis, as it makes possible to describe it in terms of a single scalar quantity L, the vortex length density (having units of (length)$^{-2}$).

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Recent experimental and numerical results lead us to consider inhomogeneous and anisotropic tangles [6–8], with special emphasis on the role of vortex diffusion. Here we will consider inhomogeneous situations, where the tangle may still be described by L, but L may change from point to point in the volume of the tangle and an evolution equation for it must be written, taking explicitly into account the contributions of inhomogeneity, either as a diffusion flux of vortices, or as an additional contribution to vortex formation or destruction. This may be especially relevant in studies about the spatial distribution of the velocity of the normal component; or in analyses of strongly inhomogeneous flows, as for instance radial flows or flows in convergent or divergent channels [9–12], or in the diffusion contribution to the decay of vortex tangles in narrow channels [13]. The description of anisotropic situations would require us to use a tensor instead of a scalar (see Ref. [14]), to take into account the different properties of vortices in the different spatial directions and we will not consider it here.

Non-equilibrium thermodynamic methods have been proposed on several occasions to describe the constitutive equations of homogeneous superfluid turbulence [15–21]. In previous papers several non-homogeneous effects were already considered in the equation for L [20,21]. Here, we reexamine the thermodynamical derivation of those equations from a more general perspective, allowing for the explicit role of inhomogeneities. Since inhomogeneities in the heat flux are expected to be deeply coupled with those in vortex density, the aim of building a model with both kinds of inhomogeneities seems well-motivated, and may play a relevant role in channels with non-homogeneous cross-section. For instance, several observations in some of these channels [10,11] show for the vortex line density some features which cannot be described by a local form of Vinen's equation, because they depend not only on the modulus of the heat flux but also on its relative orientation with respect to the direction of convergent (or divergent) cross-section. Then, the roles of ∇L and their relative direction with respect to $\mathbf{q}$, as well as that of $\nabla \mathbf{q}$ itself, should be incorporated into a wider, more inclusive formalism.

A second, more formal, aspect we discuss is the form of the entropy flux, which arises in the thermodynamic derivation of such generalized equations, and which is far more general than the classical entropy flux. A first study on this subject was made in Ref. [22], where the form of the entropy flux was analyzed in the case of a rigid heat conductor subject to heating. This quantity does not receive much interest, in usual works, but we show here that a discussion of it may be helpful to illustrate some differences between several thermodynamic formalisms, as for instance Rational Extended Thermodynamics (RET) [23] and Extended Irreversible Thermodynamics (EIT) [24], which arrive to the same final result for the transport equations but along different formal and conceptual strategies. We also compare it with the form obtained in some microscopic proposal, based on maximum-entropy methods in information theory [25,26]. In this way, the analysis of the entropy flux becomes deeply related to non-local effects and it is not an academic exercise. Since the discussions about non-classical versions of the entropy flux are very scarce in the literature, the present analysis may be useful as an explicit illustration of this quantity.

The contents are organized as follows: in Section 2 we introduce the balance equations; in Section 3 we present the results of the second-law restrictions on the constitutive equations making use of the Liu procedure. Section 4 discusses the form and role of the entropy and the entropy flux. Section 5 examines the form of the several constitutive equations. Section 6 proposes an experiment to check the most characteristic predictions of the new equations.

2. Balance equations

The basic variables we will use in this analysis are internal energy E (per unit volume), vortex length density L, and heat flux density $\mathbf{q}$. The barycentric velocity $\mathbf{v}$ is assumed to be zero (counterflow situation) and the mass density ρ is assumed to stay constant (otherwise, both ρ and $\mathbf{v}$ should be included as independent variables, as in Refs. [20,21], where a more general model, that includes nonlocal terms and ρ and $\mathbf{v}$ as further variables, is formulated).

The evolution equations are assumed to have the following general form:

$$\begin{cases} \partial_t E + \nabla \cdot \mathbf{q} = 0, \\ \partial_t \mathbf{q} + \nabla \cdot \mathbf{J}^{\mathbf{q}} = \sigma^{\mathbf{q}}, \\ \partial_t L + \nabla \cdot \mathbf{J}^L = \sigma^L, \end{cases} \tag{2.1}$$

where ∂_t indicates the time derivatives, $\mathbf{J}^{\mathbf{q}}$ being the flux of the heat flux, and $\mathbf{J}^L$ the flux of vortex lines; $\sigma^{\mathbf{q}}$ and σ^L are terms describing the net production of heat flux and of vortices per unit time and volume (though the flux of vortex lines has a direct and intuitive meaning, the flux of the heat flux is more formal and less intuitive, but it is often used in non-local formulations of heat transfer [24]; for instance, in ideal gases it is related to the fourth-order moments of the velocity distribution function with respect to the molecular velocity; in phenomenological theories, it may be often expressed in terms of the gradient of the heat flux, as it will be done in (2.5), with measurable coefficients).

We start our analysis from an entropy S and an entropy flux $\mathbf{J}^S$ of the form

$$S = S(E, L, \mathbf{q}, \nabla E, \nabla L, \nabla \mathbf{q}), \tag{2.2}$$

$$\mathbf{J}^S = \mathbf{J}^S(E, L, \mathbf{q}, \nabla E, \nabla L, \nabla \mathbf{q}). \tag{2.3}$$

The motivation to include the gradients is to describe non-local effects. In fact, we may exclude the terms in ∇E, ∇L and $\nabla \mathbf{q}$ from S, because we are not looking for an evolution equation for the gradients themselves, and it is known [27–29] that in this case the entropy does not depend on the gradients. This result allows us to simplify our presentation and to focus directly on the purely non-local contributions to the transport equations, which are known to be deeply related to