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Are power-law distributions an equilibrium distribution or a stationary nonequilibrium distribution?

Guo Ran, Du Jiulin *

Department of Physics, School of Science, Tianjin University, Tianjin 300072, China

HIGHLIGHTS

- We study whether power-law distributions satisfy the principle of detailed balance.
- With the detailed balance and generalized FDR, we derive the power-law stationary solutions of the Fokker–Planck equations.
- Power-law distributions can either be a stationary nonequilibrium distribution or an equilibrium distribution.

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ABSTRACT

We examine whether the principle of detailed balance holds for the power-law distributions that are generated from the well-known two-variable Langevin equation and the associated Fokker–Planck equations. With the detailed balance and the generalized fluctuation–dissipation relation, we derive analytically the stationary power-law distribution from the Ito's, Stratonovich's and Zwanzig's Fokker–Planck equations, and conclude that the power-law distributions can either be a stationary nonequilibrium distribution or an equilibrium distribution, which depend on information about the form of the diffusion coefficient function and the existence and uniqueness of an equilibrium state.

1. Introduction

Power-law distributions have been found frequently in some complex systems, such as the kappa-distributions measured by the observations of the solar wind and space plasmas [1–6], and lots of α-distributions noted in physics, chemistry and elsewhere like $P(E) \sim E^{-\alpha}$ with an index $\alpha > 0$ [7–11]. As the systems displaying the power-law behaviors cannot be well explained by the traditional statistical mechanics, the investigations about the physical mechanism behind the power-law distributions and their dynamical origins become increasingly attractive, and it is very important for us to understand the nature of many different processes in physical, chemical, biological, technical and their inter-disciplinary fields. Some of the power-law distributions associated with complex systems have been modeled under the framework of nonextensive statistics [12].

In the stochastic dynamical theory of power-law distributions, one needs to analyze the stationary behaviors based on the stochastic differential equations for complex dynamical processes, such as Boltzmann equations, Langevin equations and the associated Fokker–Planck (F–P) equations. Usually, it is difficult to solve a general multi-variable F–P equation for a complex system. Thus the previous works have basically focused on some of single-variable and linear F–P equations [13–22]. Most recently, many forms of the power-law distributions, including the plasma κ-distribution, the q-distribution and the

* Corresponding author.
E-mail address: jldu@tju.edu.cn (J. Du).

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α-distribution, were generated from the well-known two-variable Langevin equation, with an inhomogeneous friction and a multiplicative noise, and the associated F–P equations [23,24], which lead to find a generalized fluctuation–dissipation relations (FDR) for power-law distributions, a generalized Klein–Kramers equation and a generalized Smoluchowski equation. Based on the relevant statistical theory, one can generalize the transition state theory to the nonequilibrium systems with power-law distributions [22], one can study the mean first passage time for power-law distributions [25], the escape rate for power-law distributions in the overdamped systems [26], and the power-law reaction rate coefficient for an elementary bimolecular reaction [27]. A problem was proposed at the end of Ref. [23]: Do the power-law distributions represent a stationary nonequilibrium distribution or an equilibrium distribution? Further, how do we judge them? If they are a stationary nonequilibrium distribution, do they satisfy the condition of detailed balance? Does the conclusion have difference between the solutions of Ito's, Stratonovich's and Zwanzig's F–P equations? In this work, we will study these problems.

The paper is organized as follows. In Section 2, we briefly review the power-law distributions generated from the Langevin equation and the associated F–P equations. In Section 3, we examine whether those power-law distributions satisfy the condition of detailed balance. In Section 4, based on the detailed balance and the generalized FDR, the power-law distributions are obtained by solving the stationary F–P equations. In Section 5, we make discussions of the principle of detailed balance, an equilibrium state and a stationary nonequilibrium state. Finally in Section 6, the conclusion is presented.

2. The power-law distributions from the Langevin equations

The Langevin equations modeling the Brownian particles moving in an inhomogeneous medium with the friction coefficient, $\gamma(x,p)$ and in the potential field, $V(x)$, can be written for the position x and momentum p [23] as

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{p}{m},\qquad \frac{\mathrm{d}p}{\mathrm{d}t}=-\frac{\mathrm{d}V(x)}{\mathrm{d}x}-\gamma(x,p)p+\eta(x,p,t), \tag{1}$$

where m is the particle's mass, and $\eta(x,p,t)$ is multiplicative (space/velocity dependent) noise. As usual, the noise is assumed to be Gaussian and satisfies the zero average and the delta-correlated in time t,

$$\langle\eta(x,p,t)\rangle=0,\qquad \langle\eta(x,p,t)\eta(x,p,t')\rangle=2D(x,p)\delta(t-t'), \tag{2}$$

with the diffusion coefficient $D(x,p)$. If $\rho=\rho(x,p,t)$ is the probability distribution function with regard to coordinate x, momentum p and time t, the associated Ito's, Stratonovich's and Zwanzig's (or backward Ito's rule [28]) F–P equations can be expressed in a unified form [24] as

$$\frac{\partial\rho}{\partial t}=-\frac{p}{m}\frac{\partial\rho}{\partial x}+\frac{\partial}{\partial p}\left[\frac{\mathrm{d}V(x)}{\mathrm{d}x}+\gamma(x,p)p+\sigma\frac{\partial D(x,p)}{\partial p}\right]\rho+\frac{\partial}{\partial p}D(x,p)\frac{\partial}{\partial p}\rho, \tag{3}$$

where $\sigma=1$, 1/2 and 0 corresponds respectively to the Ito F–P equation, the Stratonovich F–P equation and the Zwanzig F–P equation. In order to show that the power-law distribution is not caused by selecting different stochastic calculus rules, we deal with the unified form of F–P equations given by Eq. (3) with all the three different stochastic calculus rules. In such a general stochastic dynamics, the power-law distributions can be generated if the friction coefficient $\gamma(x,p)$ and the diffusion coefficient $D(x,p)$ satisfy the generalized FDR [23],

$$D=\gamma m\beta^{-1}\left(1-\kappa\beta E\right), \tag{4}$$

with $\beta=1/kT$, where T is temperature, $E=V(x)+p^2/2m$ is the energy, the parameter $\kappa\neq 0$ measures a distance from equilibrium. The standard FDR, $D=m\gamma\beta^{-1}$, for an equilibrium state can be recovered in the case of $\kappa=0$. The readers can see Ref. [23] for more details about the physical explanations for the generalized FDR.

Nothing has been said about requiring that $\rho(r,p,t)$ must approach an equilibrium distribution for long time if no FDR has been invoked. Based on the generalized FDR, Eq. (4), we find that Eq. (3) has power-law stationary solutions in the following two cases:

(a) For the Zwanzig F–P equation, the stationary distribution is a function of the energy E and exactly is the power-law κ-distribution in the form [23],

$$\rho_s(E)=Z_\kappa^{-1}\left(1-\kappa\beta E\right)_+^{\frac{1}{\kappa}}, \tag{5}$$

where $Z_\kappa=\iint \mathrm{d}x\mathrm{d}p(1-\kappa\beta E)_+{}^{1/\kappa}$ is the normalization factor, and $(y)_+=y$ for $y>0$ and zero otherwise.

(b) Generally, if the friction and diffusion coefficients are both a function of the energy E, the stationary solution of Eq. (3) is exactly the power-law distribution with two parameters κ and σ in the form [24]:

$$\rho_s(E)=Z_{\sigma,\kappa}^{-1}\left(1-\kappa\beta E\right)_+^{\frac{1}{\kappa}}D^{-\sigma}, \tag{6}$$

where $Z_{\sigma,\kappa}=\iint \mathrm{d}x\mathrm{d}p(1-\kappa\beta E)^{1/\kappa}D^{-\sigma}$ is the normalization factor. Therefore we have also obtained the stationary power-law distribution from the Ito's and the Stratonovich's form of F–P equations.