Physica A 406 (2014) 73–79

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Pricing European option under the time-changed mixed Brownian-fractional Brownian model[☆]

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HIGHLIGHTS

- We proposed a mixed Brownian-fractional subdiffusive Black–Scholes model.
- This model can capture the phenomenon of long-range dependence and constant period of values.
- In a discrete time setting, we obtain the pricing formula for the European call option.

ARTICLE INFO

Article history:
Received 23 September 2012
Received in revised form 5 March 2014
Available online 20 March 2014

Keywords:
Option pricing
Mixed Brownian-fractional Brownian motion
Time-changed process

ABSTRACT

This paper deals with the problem of discrete time option pricing by a mixed Brownian-fractional subdiffusive Black–Scholes model. Under the assumption that the price of the underlying stock follows a time-changed mixed Brownian-fractional Brownian motion, we derive a pricing formula for the European call option in a discrete time setting.

1. Introduction

The classical Black–Scholes (BS) model [1] consists of three assets: a risky asset (e.g. a stock), a risk free bond with constant interest rate and the option being priced. It is assumed that the price of a stock $X(t)$ follows a geometric Brownian motion (gBm)

$$X(t) = X_0 \exp(\mu t + \sigma B(t)), \qquad X(0) = X_0 > 0, \tag{1}$$

where $B(t)$ is a standard Brownian motion, the rate of the return μ and the volatility σ are constants.

However, in the last few years, based on some empirical studies it has been shown that the BS model cannot capture many of the characteristic features of prices, such as: heavy tailed, long-range correlations and lack of scale invariance, periods of constant values, etc. Therefore, one should replace the classical BS model by other models.

Fractional BS model is an extension of the BS model, which displays the long-range dependence observed in the empirical data. (one can refer [2–5] to see more about the fractional BS model). The fractional BS model is based on the following

☆ This work is supported by the National Natural Science Foundation of China (11271153).

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http://dx.doi.org/10.1016/j.physa.2014.03.032
0378-4371/

fractional Brownian motion (fBm)

$$\tilde{X}(t) = \tilde{X_0} \exp\{\mu t + \sigma B_H(t)\}, \quad \tilde{X_0} > 0,$$

where $B_H(t)$ is a standard fractional Brownian motion with Hurst exponent $H \in (\frac{1}{2}, 1)$. Unfortunately, due to fBm is neither a Markov process nor a semi-martingale, it is have been shown [6,7] that the fractional BS model admits arbitrage in a complete and frictionless market. To get around this problem and to take into account the long memory property, it has been suggested [8,9] that it is reasonable to use the mixed Brownian-fractional Brownian motion (mBfBm) to capture the fluctuations of financial asset. Now, we recall the definition of the mfBm [10].

A mixed Brownian-fractional Brownian motion of parameters a, b and H is a process $M^H = \{M_t^H(a, b); t \geq 0\} = \{M_t^H; t \geq 0\}$, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ by

$$\forall t \in R_+, \quad M_t^H = M_t^H(a, b) = aB(t) + bB_H(t),$$

where $B(t)_{t\in R_+}$ is a Brownian motion, and $B_H(t)_{t\in R_+}$ is an independent fractional Brownian motion of Hurst parameter H. The mixed Brownian-fractional Brownian model is based on the mBfBm, i.e.

$$\bar{X}(t) = \bar{X}_0 \exp\{\mu dt + \sigma M_t^H\},$$

in the above mixed model with $a \neq 0$ and $b \neq 0$ some results were obtained. Kuznetsov [11] proved the absence of arbitrage under condition of independence of processes $B(t)$ and $B_H(t)$. Cheridito [12] proved that, for $H \in (\frac{3}{4}, 1)$, the mixed model is equivalent to the Brownian motion and hence it is arbitrage-free. For $H \in (\frac{1}{2}, 1)$, Mishura and Valkeila [13] proved that the mixed model is arbitrage-free. To see more about the mixed model, one can refer to Refs [11,14,12,13].

In order to describe properly financial data exhibiting periods of constant values, Magdziarz [15] introduced subdiffusive geometric Brownian motion (sgBm)

$$X_\alpha(t) = X(T_\alpha(t)),$$

as the model of the asset price. Here the parent process $X(\tau)$ is the gBm defined in (1), $T_\alpha(t)$ is the inverse α-stable subordinator with parameter $\alpha \in (0, 1)$. Furthermore, $T_\alpha(t)$ is assumed to be independent of the Brownian motion $B(t)$. Magdziarz pointed out that this model is arbitrage free but incomplete, and based on the sgBm he obtained the corresponding subdiffusive BS formula for the fair price of European options.

Subdiffusion is a well known and established phenomenon in statistical physics. The usual model of subdiffusion in physics is developed in terms of FFPE (fractional Fokker–Planck equations). This equation was first derived from the continuous-time random walk scheme with heavy-tailed waiting times [16]. It provides a useful way for the description of transport dynamics in complex systems [17]. Another description of subdiffusion is in terms of subordination, where the standard diffusion process is time-changed by the so-called inverse subordinator[17,18]. Following this line, in this paper, we introduce a time-changed mixed Brownian fractional BS model, whose price of the underlying stock is

$$S_t = \hat{X}(T_\alpha(t)) = S_0 \exp\{\mu T_\alpha(t) + \sigma(aB(T_\alpha(t)) + bB_H(T_\alpha(t)))\}, \quad S_0 = \hat{X}(0) > 0. \tag{2}$$

This model not only exhibits the long-range dependence property, but also can take into account financial data exhibiting periods of constant values.

When the price of the underlying stock S_t satisfies Eq. (2), we derive an explicit option pricing formula for the European call option. This formula is similar to the Black–Scholes option pricing formula, but with the volatility being different. The modified volatility $\hat{\sigma}^2 = \sigma^2 W_{\alpha,H}(t, \Delta t)\Delta t^{-1}$, σ is the volatility used in Black–Scholes option pricing formula, $W_{\alpha,H}(t, \Delta t)$ denotes the second moments of the random variable $\Delta M_{\alpha,H}(t)$. In Remark 3.2, we will show how to obtain the value of $\hat{\sigma}$.

The remainder of this paper is organized as follows. In Section 2, we introduce some properties of the time-changed mixed Brownian-fractional BS model. In Section 3, we investigate the option pricing in a discrete time setting and obtain a formula for the European call option. In Section 4, we conclude this paper.

2. Time-changed mBfBm

The time-changed process $T_\alpha(t)$ is the inverse α-stable subordinator defined as below

$$T_\alpha(t) = \inf\{\tau > 0: U_\alpha(\tau) > t\},$$

here $U_\alpha(\tau)_{\tau\geq 0}$ is a strictly increasing α-stable Lévy process [19,20] with Laplace transform: $\mathbb{E}(e^{-uU_\alpha(\tau)}) = e^{-\tau u^\alpha}$, $\alpha \in (0, 1)$. $U_\alpha(t)$ is $\frac{1}{\alpha}$ self-similar and $T_\alpha(t)$ is α self-similar in the sense that $U_\alpha(t)$ has the same law as $c^{\frac{1}{\alpha}} U_\alpha(t)$ and $T_\alpha(t)$ has the same law as $c^\alpha T_\alpha(t)$. Specially, when $\alpha \uparrow 1$, $T_\alpha(t)$ reduces to the "objective time" t. One can refer [21,22] to see more details about the subordinator and its inverse.

Consider the subordinated process $M_{\alpha,H}(t) = M^H_{T_\alpha(t)} = aB(T_\alpha(t)) + bB_H(T_\alpha(t))$, here the parent process M_t^H is a mBfBm and $T_\alpha(t)$ is assumed to be independent of M_t^H. We call M_t^H a subdiffusion process. When $a = 0$, $b = 1$, it is a subdiffusion process considered in Ref. [23]. Specially, when $a = 1$, $b = 1$ and $\alpha = 1$, then it is the process considered in Ref. [14].

For $\beta > 0$, a random function $h(x)$ is said to be $o(x^\beta)$ if $\lim_{x\downarrow 0} \frac{\mathbb{E}(|h(x)|^n)}{x^{n\beta}} = 0$ for every $n \in \mathbb{N}$.