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# On the axiomatic requirement of range to measure uncertainty

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## HIGHLIGHTS

- This manuscript studies the uncertainty measure in Dempster–Shafer theory.
- The irrationality of the axiomatic requirement of range has been pointed out.
- The correct range of uncertainty is $[0, \log_2 2^{|X|}]$ rather than $[0, \log_2 |X|]$.

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ABSTRACT

How to measure uncertainty is still an open issue. Probability theory is a primary tool to express the aleatoric uncertainty. The Shannon's information entropy is an effective measure for the uncertainty in probability theory. Dempster–Shafer theory, an extension of probability theory, has the ability to express the aleatoric and epistemic uncertainty, simultaneously. With respect to such uncertainties in Dempster–Shafer theory, a justifiable uncertainty measure is required to satisfy five axiomatic requirements based on previous studies. In this paper, we show that one of the axiomatic requirements, the requirement of *range*, is questionable. The correct range of uncertainty should be $[0, \log_2 2^{|X|}]$ rather than $[0, \log_2 |X|]$ according to the concept of entropy.



## 1. Introduction

Uncertainty is ubiquitous in nature. How to measure the uncertainty has attracted much attention [1–3]. Various categorizations exist to accommodate different kinds of uncertainties. An existing classification scheme [4] divides the uncertainties into five types: (i) aleatoric uncertainty which mainly comes from random or stochastic processes; (ii) epistemic uncertainty which is due to the lack of knowledge; (iii) irreducible uncertainty which is a natural variability that cannot be reduced but only quantified; (iv) reducible uncertainty that is from the lack of specific information, knowledge and can be reduced with acquisition of more information; (v) inference uncertainty. Numerous uncertainty theories have been developed, such as probability theory [5], fuzzy set theory [6], possibility theory [7], Dempster–Shafer theory [8,9], and random intervals [10].

Since first proposed by Clausius in 1865 for thermodynamics [11], the concept of entropy has emerged in a large number of fields and has been a measure of disorder and uncertainty [12–21]. Information entropy [22], derived from the Boltzmann–Gibbs (BG) entropy [23] in thermodynamics and statistical mechanics, has been an indicator to measure

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aleatoric uncertainty which is associated with the probability density function (PDF) or probability mass function (PMF). For the aleatoric uncertain information expressed by PDF or PMF, information entropy is a good measure. However, with respect to other uncertain information including epistemic, irreducible, reducible and inferential uncertainty, classical information entropy is invalid. In this paper, one of these uncertainties, epistemic uncertainty, is taken into consideration. Dempster–Shafer theory [8,9] is mainly proposed to handle such uncertainty. In Dempster–Shafer theory, the epistemic uncertainty simultaneously contains nonspecificity and discord [24] which are coexisting in a basic probability assignment function (BPA). Several uncertainty measures, such as AU [25,26], AM [24], have been proposed to quantify such uncertainty in Dempster–Shafer theory. What is more, five axiomatic requirements have been further built in order to develop a justifiable measure. These five axiomatic requirements are range, probabilistic consistency, set consistency, additivity, and subadditivity, respectively [27].

These axiomatic requirements greatly promote the study of epistemic uncertainty measure to a large extent. However, it has some shortcomings. The most controversial point is the axiomatic requirement of range. According to the axiomatic requirement of range, the range of a justifiable measure is limited in $[0, \log_2 |X|]$, where $|X|$ is the cardinality of frame of discernment $X$. But it is not reasonable. In this paper, we will show its irrationality in terms of the basic concept of entropy. The correct range should be $[0, \log_2 2^{|X|}]$.

In what follows, some background knowledge about Dempster–Shafer theory, entropy and the axiomatic requirement of range in uncertainty measure are briefly introduced first. Then, the irrationality of the axiomatic requirement of range is discussed. Finally, this paper is concluded.

## 2. Dempster–Shafer theory

Dempster–Shafer theory (short for D–S theory), also called belief function theory, as introduced by Dempster [8] and then developed by Shafer [9], has emerged from their works on statistical inference and uncertain reasoning. This theory is widely applied to uncertainty modeling [28–34]. D–S theory mainly focus on the epistemic uncertainty, but it is also valid for aleatoric uncertainty. It has many merits by contrast probability theory. First, D–S theory can handle more uncertainty in real world. In contrast to the probability theory in which probability masses can be only assigned to singleton subsets, in D–S theory the belief can be assigned to both singletons and compound sets. Second, in D–S theory, prior distribution is not needed before the combination of information from individual information sources. Third, D–S theory allows one to specify a degree of ignorance in some situations instead of being forced to be assigned for probabilities. Some basic concepts in D–S theory are introduced.

Let $X$ be a set of mutually exclusive and collectively exhaustive events, indicated by

$$X = \{\theta_1, \theta_2, \ldots, \theta_i, \ldots, \theta_{|X|}\} \tag{1}$$

where set $X$ is called a frame of discernment. The power set of $X$ is indicated by $2^X$, namely

$$2^X = \{\emptyset, \{\theta_1\}, \ldots, \{\theta_{|X|}\}, \{\theta_1, \theta_2\}, \ldots, \{\theta_1, \theta_2, \ldots, \theta_i\}, \ldots, X\}. \tag{2}$$

For a frame of discernment $X = \{\theta_1, \theta_2, \ldots, \theta_{|X|}\}$, a mass function is a mapping $m$ from $2^X$ to $[0, 1]$, formally defined by:

$$m : 2^X \to [0, 1] \tag{3}$$

which satisfies the following condition:

$$m(\emptyset) = 0 \quad \text{and} \quad \sum_{A \in 2^X} m(A) = 1. \tag{4}$$

In D–S theory, a mass function is also called a basic probability assignment (BPA). Assume there are two BPAs indicated by $m_1$ and $m_2$, the Dempster's rule of combination is used to combine them as follows:

$$m(A) = \begin{cases} \dfrac{1}{1-K} \displaystyle\sum_{B \cap C = A} m_1(B) m_2(C), & A \neq \emptyset; \\ 0, & A = \emptyset, \end{cases} \tag{5}$$

with

$$K = \sum_{B \cap C = \emptyset} m_1(B) m_2(C). \tag{6}$$

Note that, the Dempster's rule of combination is only applicable to such two BPAs which satisfy the condition $K < 1$.

D–S theory has more advantages in handling uncertainty compared to the classical probability theory [35–39]. When information is adequate, probability theory is effective to handle that situation. However, when information is not adequate, probability theory is invalid to such uncertain situation. Here is an example.

Assume that there are two boxes, as shown in Fig. 1. In the left box, there are only red balls. In the right box, there are not only red balls but also green balls. However, the exact number of red balls, green balls, and the ratio of them are completely