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A theoretical characterization of scaling properties in a bouncing ball system

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HIGHLIGHTS

- We characterized an analytical approach for scaling laws.
- A two-dimensional nonlinear mapping was set as the model under study.
- Some phase transitions were characterized.
- Scaling laws were obtained and characterized for the numerical results.

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ABSTRACT

Analytical arguments are used to describe the behavior of the average velocity in the problem of an ensemble of particles bouncing a heavy and periodically moving platform. The dynamics of the system is described by using a two-dimensional mapping for the variables' velocity and discrete time n. In the absence of dissipation and depending on the control parameter and initial conditions, diffusion in energy is observed. Considering the introduction of dissipation via inelastic collisions, we prove that the diffusion is interrupted and a transition from unlimited to limited energy growth is characterized. Our result is general and can be used when the initial condition is a very low velocity leading to a growth of average velocity with $\sqrt{n}$ or for large initial velocity where an exponential decay of the average velocity is observed. The results obtained generalize the scaling observed in the bouncer model as well as the stochastic and dissipative Fermi–Ulam model. The formalism can be extended to many other different types of models, including a class of time-dependent billiards.

1. Introduction

Phase transitions in statistical mechanics have a common and hence important feature called as scale invariance [1,2]. Near a phase transition, physical observables may fluctuate at different length scales leading them to be described by a homogeneous generalized function [3]. The formalism leads also to a set of critical exponents [4] that describe well the behavior of physical observables near a phase transition. Often it is possible to obtain a relation of the exponents defining a scaling law. It is said for different systems having the same set of critical exponents to belong to the same class of

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universality [5]. From statistical mechanics standpoint, a phase transition is related to abrupt changes in the spatial structure of the system mainly due to changes in the control parameter, while for a dynamical system a phase transition is linked particularly to modifications in the structure of the phase space of the system. Near the transition, the dynamics of the system can be described using a scaling function and critical exponents to characterize the dynamics.

In the majority of the cases, dynamical systems are described by differential equations. When symmetries and conserved quantities are present, the solutions of the differential equations can be qualitatively modeled by nonlinear mappings [6]. Dissipation in mappings is a quite common task [7–14] and transform the mixed structure of the conservative cases into a set of attractors that can be chaotic or simply periodic.

In this paper we apply scaling formalism to explore the dynamics of an ensemble of classical particles experiencing collisions with an infinitely heavy and periodically time dependent platform in the presence of a constant gravitational field. Collisions of the particles with the moving wall can be considered as: (a) inelastic leading to a fractional loss of energy upon each collision or; (b) as elastic. The dynamics of the model is described by using a two-dimensional, nonlinear and area contracting mapping for the variables' velocity and phase–time at the instant of the impact. Our major contribution in this paper is to describe and hence predict, by means of analytical arguments, the behavior of the average velocity for an ensemble of particles either if the initial velocity is: (i) small or; (ii) large as compared to the velocity of the moving platform. These two points together with the formalism describing a scaling on the average velocity as function of the control parameters are what give the originality of the paper and the major contribution to the area. If the initial velocity is small (case (i)), an initial growth of the average velocity is observed and scales with $\sqrt{n}$ until reaching a regime of constant velocity for large enough n. So far the results known in the literature [15,16] (and references cited therein) were made by considering *scaling hypotheses* for short and long enough time, leading to critical exponents, obtained by extensive numerical simulations. Our approach in this paper generalizes the previous results known in the literature without the need of proposing scaling hypotheses. On the other hand for case (ii), our result predicts an exponential decay in the velocity that is remarkably well supported by numerical simulations. Both cases (i) and (ii) are supported by numerical simulations. As we shall see, the formalism used here can be extended to many other different types of problems. Immediate applications can be made to a class of problems called billiard type systems.

2. The model and procedure

The dynamics of the system [17] is given by a mapping $T(V_n, \phi_n) = (V_{n+1}, \phi_{n+1})$, where V and ϕ are the velocity of the particle and phase of the moving wall respectively at the instant of the nth collision. The position of the moving platform is given by $y_w = \epsilon \cos(\omega t)$, where ϵ is the amplitude of the oscillation and ω is the angular frequency. The dynamics is better described [15,16] by considering a set of dimensionless variables namely $V_n = v_n\omega/g$, $\varepsilon = \epsilon\omega^2/g$ while the time is measured as $\phi_n = \omega t_n$. Notice that the control parameter ε furnishes the ration of the acceleration, namely, the maximum acceleration of the moving wall by the gravitational field. It characterizes a transition from integrability ($\varepsilon = 0$) to non-integrability ($\varepsilon \neq 0$). When the amplitude of oscillation is small enough, the dynamics of the system can be described by using a mapping [17] of the type

$$T : \begin{cases} V_{n+1} = |\gamma V_n - (1+\gamma)\varepsilon \sin(\phi_{n+1})| \\ \phi_{n+1} = [\phi_n + 2V_n] \bmod (2\pi). \end{cases} \quad (1)$$

The modulo considered in the first equation of mapping (1) is used to avoid the particle moving beyond the wall after a collision. The mapping is area contracting because the determinant of the Jacobian matrix is given by

$$\mathrm{Det}J = \gamma^2 \mathrm{sign}[\gamma V_n - (1+\gamma)\varepsilon \sin(\phi_{n+1})], \quad (2)$$

where $\mathrm{sign}(z)$ is 1 for $z > 0$ and -1 for $z < 0$.

According to the results of Lichtenberg and Lieberman [17], for the control parameter $\varepsilon > \varepsilon_c = 0.2429\ldots$, the conservative dynamics obtained with $\gamma = 1$ leads to unlimited diffusion in velocity [18,19]. This is observed because the system experiences a transition from local to global chaotic behavior. Therefore the invariant spanning curves observed in the phase space for $\varepsilon < \varepsilon_c$ are destroyed leading to the evolution of some initial conditions to exhibit unlimited diffusion [19]. On the other hand when $\varepsilon > \varepsilon_c$ and $\gamma < 1$, the unlimited diffusion of the velocity is terminated. Indeed since there is area contraction (see the expression of the Jacobian matrix), attractors in the phase space appear leading to a suppression of the unlimited energy growth of a bouncing particle. This is the property we shall explore now. Fig. 1 shows a plot of the phase space for: the conservative case $\gamma = 1$ and (a) $\varepsilon = 0.2$ and (b) $\varepsilon = 0.3$ and dissipative case considering an ensemble of 10^3 different initial phases $\phi_0 \in [0, 2\pi]$ for a fixed initial velocity $V_0 = 1.0$ for (c) $\varepsilon = 10$ and $\gamma = 0.99$.

We see that the conservative case exhibits a set of periodic islands which are not observed in the non-dissipative case. When dissipation is introduced, the stability islands are destroyed, leading the system to present sinks and a chaotic attractor. Also, the system is very sensitive to the variation of the control parameter ε. According to Refs. [20,21], for small values of ε, the system can present several different sinks and attractors, leading the dynamics to obey much more complex behavior for long time evolution. As we increase ε, the basins of attraction of these sinks are destroyed, leaving only the chaotic attractor present in the dynamics. Because of that, we are only dealing in the scaling analysis with a higher regime of ε. In this regime, since $\gamma < 1$, the average velocity converges to a constant plateau at large enough time (number of