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Algebraic connectivity of interdependent networks

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HIGHLIGHTS

- We prove that the spectra of interdependent graphs experience a phase transition.
- The transition is characterized by the saturation of diffusion processes.
- An expression for the transition is found as a function of a coupling constant.
- The transition point depends on the network types and the link addition strategy.

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ABSTRACT

The algebraic connectivity μ_{N-1}, i.e. the second smallest eigenvalue of the Laplacian matrix, plays a crucial role in dynamic phenomena such as diffusion processes, synchronization stability, and network robustness. In this work we study the algebraic connectivity in the general context of interdependent networks, or network-of-networks (NoN). The present work shows, both analytically and numerically, how the algebraic connectivity of NoNs experiences a transition. The transition is characterized by a saturation of the algebraic connectivity upon the addition of sufficient coupling links (between the two individual networks of a NoN). In practical terms, this shows that NoN topologies require only a fraction of coupling links in order to achieve optimal diffusivity. Furthermore, we observe a footprint of the transition on the properties of Fiedler's spectral bisection.

1. Introduction

In the last decades, there has been a significant advance in understanding the structure and functioning of complex networks [1,2]. Statistical models of networks are now widely used to describe a broad range of complex systems, from networks of human contacts to interactions amongst proteins. In particular, emerging phenomena of a population of dynamically interacting units has always fascinated scientists. Dynamic phenomena are ubiquitous in nature and play a key role within various contexts in sociology [3], and technology [4]. To date, the problem of how the structural properties of a network influences the convergence and stability of its synchronized states has been extensively investigated and discussed, both numerically and theoretically [5–9], with special attention given to networks of coupled oscillators [10–13].

In the present work, we focus on the second smallest eigenvalue μ_{N-1} of a graph's Laplacian matrix, also called *algebraic connectivity*. This metric plays an important role on, among others, synchronization of coupled oscillators, network robustness, consensus problems, belief propagation, graph partitioning, and distributed filtering in sensor networks [14–18]. For example, the time it takes to synchronize Kuramoto oscillators upon any network scales with the inverse of μ_{N-1} [19–22]. In other words, larger values of μ_{N-1} enable synchronization in both discrete and continuous-time systems, even in the presence of transmission delays [23,24]. As a second application, graphs with "small" algebraic connectivity

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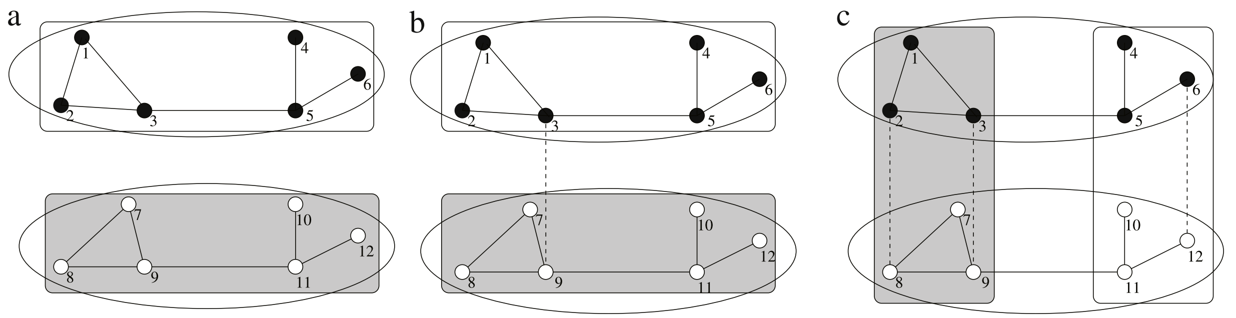

Fig. 1. Two graphs, with 6 nodes, and 7 links each (solid lines) are subject to spectral partitioning based on the Fiedler vector, i.e. the eigenvector corresponding to the algebraic connectivity. The two single graphs are encircled with ovals, and the corresponding spectral partitioning is represented by rectangles. The single graphs are progressively coupled with (a) no interlinks, (b) 1 interlink and (c) 3 interlinks (dashed lines). As we can see, adding 1 interlink causes the spectral partitioning algorithm to split the network into the two original partitions. Upon the addition of 3 or more interlinks, the spectral partition experiences a brusque transition, causing the single networks intralinks (solid lines) to become the new confining links.

have a relatively clean bisection, i.e. the smaller μ_{N-1}, the fewer links must be removed to generate a bipartition [25]. Furthermore, we illustrate the role of the algebraic connectivity in the diffusion dynamic process. For the sake of simplicity, we model the diffusive dynamics as a commodity exchange governed by the following differential equation:

$$\frac{ds_i}{dt} \approx \sum_{j \in N_i} (s_i(t) - s_j(t)) \approx \sum_{j=1}^{N} Q_{ij} s_j(t); \qquad (1)$$

where s_i represents the commodity or the state of the ith component, N_i its neighbors, and Q the Laplacian matrix, as further defined in Section 2. The equilibrium state is that in which all gradients in (1) reach zero, thus the rate of the slowest exponential decay (of the deviation from the equilibrium) is proportional to the algebraic connectivity [26]. Hence, the higher the algebraic connectivity of the Q matrix, the smaller the "proper time".

Despite the latest advances in the research on synchronization and graph spectra, current research methods mostly focus on individual networks treated as isolated systems. In reality, complex systems are seldom isolated. For example, a power grid and a communication network may strongly depend on each other. A power station depends on a communication node for information, whereas a communication node depends on a power station for electricity [20]; similarly, a pathogen may spread from one species to another. Much effort has been devoted to predict cascading effects in such interdependent networks [27–29]: the largest connected component has been shown to exhibit a spectacular phase transition after a critical number of faults is reached. Quite recently, a novel approach has been introduced by resorting to the spectral analysis of interdependent networks. By means of the graph spectra, the epidemic thresholds of interdependent networks have been estimated, and absolute boundaries have been provided [27]. These scenarios motivated us to study the influence of interdependent networks on diffusive processes via their spectral properties.

In this work, we show analytically and numerically how the algebraic connectivity of interdependent networks experiences a phase transition upon the addition (or removal) of a sufficient number of interlinks between two identical networks. As a direct consequence, the proper time of a diffusion process on top of the NoN system is not affected by interlink additions, as long as the number of interlinks is higher than a critical threshold. The location of the described transition depends on the link addition strategy, as well as on the algebraic connectivity of the single networks. Gomez et al. [30] applied perturbation theory to approximate lower bounds for μ_{N-1} in a multiplex scenario, for which they conclude that interdependent networks speed up diffusion processes. Although the latter study hints the existence of a sudden shift, their authors over-sighted the existence of a phase transition, which we fully characterize via mean-field theory and by investigating additional spectral properties for different values of N. In particular, we observe that the phase transition is reflected in spectral partitioning algorithms, as illustrated in Fig. 1.

This paper is structured as follows. Section 2 introduces some required terminology, the Laplacian matrix, and its corresponding spectra. Sections 3.1 and 3.2 provide some analytical results for the algebraic connectivity of interdependent networks, based on both mean-field approach and perturbation theory, respectively. Our models are able to predict the fraction of links that will cause the algebraic connectivity transition. Finally, Section 4 validates our previous results through extensive numerical results. This section also exposes results on regular, random, small-world, and scale-free networks. Conclusions are drawn in Section 5.

2. Definitions

2.1. Graph theory basics

A graph G is composed by a set of *nodes* interconnected by a set of *links* $G(\mathcal{N}, \mathcal{L})$. Suppose one has two networks $G_1 = (\mathcal{N}_1, \mathcal{L}_1)$ and $G_2 = (\mathcal{N}_2, \mathcal{L}_2)$, each with a set of nodes $(\mathcal{N}_1, \mathcal{N}_2)$ and a set of links $(\mathcal{L}_1, \mathcal{L}_2)$ respectively. For simplicity, in the following we will suppose any dependence relation to be symmetric, i.e. all networks are undirected.