Physica A 404 (2014) 34–46

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

An entropic analysis of approximate quantum error correction

Carlo Cafaro [a,b,*], Peter van Loock [b]

[a] *Max-Planck Institute for the Science of Light, 91058 Erlangen, Germany*

[b] *Institute of Physics, Johannes-Gutenberg University Mainz, 55128 Mainz, Germany*

HIGHLIGHTS

- Entropic methods in exact and approximate quantum error correction.
- Landauer's erasure principle, quantum error correction, Maxwell demon, and the Second Law of thermodynamics.
- Links between quantum state discrimination, quantum information gain, and quantum error correction.

ARTICLE INFO

Article history:
Received 29 October 2013
Received in revised form 1 February 2014
Available online 1 March 2014

Keywords:
Entropy
Quantum error correction
Thermodynamics

ABSTRACT

The concept of entropy and the correct application of the Second Law of thermodynamics are essential in order to understand the reason why quantum error correction is thermodynamically possible and no violation of the Second Law occurs during its execution.

We report in this work our first steps towards an entropic analysis extended to approximate quantum error correction (QEC). Special emphasis is devoted to the link among quantum state discrimination (QSD), quantum information gain, and quantum error correction in both the exact and approximate QEC scenarios.

1. Introduction

It is well-established that the notion of entropy plays a key role in the foundations of quantum theory [1–6] whose statistical nature is evident when dealing with incomplete information gathered in quantum measurements. Incomplete information refers to the fact that in quantum physics, as opposed to classical physics, two non-commuting observables do not have any definite values simultaneously and therefore one cannot obtain simultaneously perfect information about both. The quantum mechanical perfect information gain always refers only to a complete set of commuting observables. In fact, combining this aspect of quantum mechanics with the notion of entanglement and nonlocality made Einstein, Podolsky, and Rosen conclude that quantum mechanics is incomplete [7]. In general, measurements are performed to increase information about physical systems. This information, if appropriate, may in principle be used for a reduction of the thermodynamical entropy of such physical systems.

In his 1929 seminal paper [8], Szilard presented the so-called Szilard's engine to show that additional information about a system yields a decrease in the entropy of the system. Szilard reaffirmed his belief in the Second Law of thermodynamics and that the measurement process performed by some sort of intelligent being (Maxwell's demon), in some overall sense,

---

[*] Corresponding author at: Max-Planck Institute for the Science of Light, 91058 Erlangen, Germany.
E-mail address: carlocafaro2000@yahoo.it (C. Cafaro).

http://dx.doi.org/10.1016/j.physa.2014.02.070
0378-4371/

requires energy dissipation. Szilard, however, did not pin down the exact source of dissipation within a measurement cycle. In 1961, Landauer showed that any erasure of information is accompanied by an appropriate increase in entropy [9]. In 1982, relying on Landauer's key observations, Bennett exorcised Maxwell's demon in a Szilard-like set-up [10]. Bennett's main conclusion was that the increase in entropy is not necessarily a consequence of observations made by the demon, but accompanies the resetting of the final state of the demon to be able to start a new cycle. In other words, information gained has to eventually be erased, which leads to an increase of entropy in the environment and prevents the Second Law of thermodynamics from being violated. In fact, the entropy increase in erasure has to be at least as large as the initial information gain. Bennett's analysis was, however, completely classical. In 1984, Zurek analyzed the demon quantum mechanically confirming Bennett's results [11].

Perhaps, quantum error correction (QEC) is the best arena for considering the links between entropy, information, and thermodynamics. A QEC technique consists in encoding quantum information into a physical system in such a way that it can be either actively or passively saved from decoherence [12]. Furthermore, since the process of quantum measurement cannot perfectly discriminate among non-orthogonal states, the optimal strategy to encode information is to prepare the d-level quantum system in one out of d orthogonal states.

In this article, we discuss an additional application of Landauer's erasure principle to show that quantum error correction, regarded as a Maxwell demon, does not violate the Second Law of thermodynamics. The main initial motivation for this work was the will of gaining a better understanding of the following statement that appeared in Ref. [13]: Doing perfect error correction without perfect information gain is forbidden by the Second Law of thermodynamics via Landauer's principle. This is analogous to von Neumann's (1952) proof that being able to distinguish perfectly between two non-orthogonal states would lead directly to the violation of the Second Law of thermodynamics.

The layout of this work is as follows. In Section 2, we mention the main historical objections to Landauer's principle which plays a key role in the comprehension of the reason why QEC is compatible with the Second Law. In Section 3, we reconsider the standard entropic analysis of a QEC cycle showing the compatibility of QEC with the Second Law. In Section 4, we specify the meaning of exact- and approximate-QEC. Motivated by the aim of a better understanding of Vedral's above-mentioned statement, in Section 5 we discuss the possibility of approximate-QEC where only an imperfect discrimination of non-orthogonal quantum states is permitted and underline some consequences of the presence of non-orthogonal quantum states in the entropic analysis of a QEC cycle. Our final remarks appear in Section 6.

2. Brief historical background

In his 1961 classic paper [9], Landauer discussed the limitation of the efficiency of computers imposed by physical laws. In particular, he provided key arguments to solve Maxwell's demon puzzle in Szilard's engine. Landauer's principle of information erasure states that when erasing one bit of information stored in a memory device, on average, at least $k_B T \log 2$ energy in the form of heat is dissipated into the environment. The quantity k_B denotes Boltzmann's constant while T is the temperature of the environment at which one erases. We stress that implicit in Landauer's argument is the crucial assumption that information entropy translates into thermodynamical entropy. Landauer's principle received several objections:

- The identification of information entropy with thermodynamical entropy is unfounded [14]. In particular, information gain should not be identified with entropy decrease.
- Landauer's claim is based only on the Second Law of thermodynamics and, although plausible, not very rigorous. For instance, piston fluctuations should be taken into consideration since they are of crucial importance in the analysis of a Szilard engine [15].
- Landauer's principle has no general validity since there exists a superconducting logic device (the so-called quantum flux parametron) capable of carrying out logically irreversible operations (information destruction, for instance) without requiring any minimal dissipation per step [16].

All these objections have been one by one rebutted to a certain extent. For instance, the first objection was rebutted by Costa de Beauregard and Tribus [17]. They stress that the concept of entropy in statistical mechanics can be deduced from the concept of information. The first objection was also reconsidered later by Peres [18,19] who, relying on previous works of von Neumann [20] (in 1952 von Neumann showed that allowing for the possibility of distinguishing perfectly two non-orthogonal quantum states would lead directly to the violation of the Second Law) and Partovi [21] (thermodynamic behavior is already present at the quantum level and is not the exclusive domain of macroscopic systems), concludes that there should be no doubt that entropy, as defined by von Neumann in quantum theory and by Shannon [22] in information theory is fully equivalent to that of classical thermodynamics. However, we remark that while entropy is measured in units of bits in classical information theory, it is measured in units of joules/kelvin in classical thermodynamics. This statement, however, he emphasizes, must be understood with the same vague meaning as when we say that quantum notions of energy, momentum, angular momentum, etc. are equivalent to the classical notions bearing the same names. The second objection was addressed by Zurek [11] who refined Szilard's analysis by taking into consideration fully quantum aspects of Slizard's engine. Finally, the third objection was considered by Landauer himself [23] who stated that what was actually showed by Goto and coworkers in Ref. [16] is that there is no minimal dissipation per step for logically reversible operations and that this, in turn, does not contradict his principle.