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Network skeleton for synchronization: Identifying redundant connections

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HIGHLIGHTS

- We find that many links in networks are actually redundant for synchronization.
- The homogeneous networks have more redundant links than the heterogeneous networks.
- The synchronization backbone is the minimal network to preserve synchronizability.
- The results are confirmed by the Kuramoto model.

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ABSTRACT

Synchronization is an important dynamical process on complex networks with wide applications. In this paper, we design a greedy link removal algorithm and find that many links in networks are actually redundant for synchronization, i.e. the synchronizability of the network is hardly affected if these links are removed. Our analysis shows that homogeneous networks generally have more redundant links than heterogeneous networks. We denote the reduced network with the minimum number of links to preserve synchronizability (eigenratio of the Laplacian matrix) of the original network as the synchronization backbone. Simulating the Kuramoto model, we confirm that the network synchronizability is effectively preserved in the backbone. Moreover, the topological properties of the original network and backbone are compared in detail.

1. Introduction

Understanding the relation between network structure and dynamical processes is one of the most significant topics in network researches [1]. Synchronization is a typical collective process in many different fields including biology, physics, engineering, and even sociology. It is known that synchronization is rooted in human life from the metabolic processes in our cells to the highest cognitive tasks we perform as a group of individuals. Since the small-world and scale-free networks were proposed, the synchronization on complex networks has been intensively investigated [2–7]. Many of these studies are based on the analysis of Master Stability Function which allows us to use the eigenratio $R = \lambda_2/\lambda_N$ of the Laplacian matrix to represent the synchronizability of a network [8,9]. The larger the eigenratio is, the stronger the synchronizability of the network is.

In the literature, many works focus on investigating how the network structure affects the synchronizability [10–14]. Since the network synchronizability can be significantly influenced by network structure, researchers proposed many methods to modify the topology of networks to enhance the synchronizability. For instance, Refs. [15,16] improve the synchronizability by adding, removing and rewiring links in networks. The synchronizability is also enhanced by manipulating links

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based on evolutionary optimization algorithms [17–19]. Additionally, the synchronizability can be improved by properly assigning link directions [9,20]. Weighting links is another way to improve the network synchronizability [3,21,22]. For instance, researchers design some strategies to make the strength of nodes homogeneous and reduce the feedback loop [2]. Through these link weighting methods, it is finally found that the directed spanning tree for any connected initial network configuration can achieve the strongest synchronizability (i.e., $\lambda_2 = \lambda_N = 1$ and $R = 1$) [23,24]. In Ref. [25], it is further demonstrated that spanning trees are not the only optimal networks for synchronization. Very recently, it is revealed that the transition from disorder state to synchronization can be the first order if the natural frequency of the oscillators are positively correlated with node degree [26].

Instead of improving the network synchronizability, we focus in this paper on investigating the redundant links for synchronization. The network synchronizability will stay almost unchanged if these links are removed. In order to accurately identify and remove the redundant links, we design a greedy algorithm based on the eigenvector of the Laplacian matrix. We validate the algorithm by directly comparing the simulation of the Kuramoto model on the original network and the reduced network [27]. With this algorithm, we find that the homogeneous networks generally have more redundant links than the heterogeneous networks do.

The above phenomenon indicates that there is a backbone structure for the network synchronization. Generally, a backbone should preserve the topological properties or the function of the original networks. For example, the degree distribution [28], betweenness [29] and transportation ability [30] and even the ability for information filtering [31] can be preserved in network backbones. More recently, a network backbone detection method based on link salience is proposed [32]. In this paper, we remove the redundant links until the synchronizability is reduced to 95% of the original one. The obtained network is considered as the synchronization backbone. The topological properties of the original network and backbone are compared in detail. Finally, we remark that this work is also meaningful from a practical point of view. Since the backbone has almost the same synchronizability of the original network but has 30% fewer links, the construction cost of real systems can be considerably reduced by excluding the links outside the backbone structure.

2. Redundant links for synchronizability in complex networks

In a dynamical network, each node represents an oscillator and the edges represent the couplings between the nodes. For a network of N linearly coupled identical oscillators, the dynamical equation of each oscillator can be written as

$$\dot{x}_i = F(x_i) - \sigma \sum_{j=1}^{N} G_{ij} H(x_j), \quad i = 1, 2, \ldots, N. \tag{1}$$

where x is the phase of each oscillator, $F(x)$ accounts for the internal dynamics of each node, $H(x)$ is a coupling function, and σ is the coupling strength. The previous works based on the analysis of Master Stability Function show that the network synchronizability can be measured by eigenratio $R = \lambda_2/\lambda_N$ where λ_2 and λ_N are respectively the smallest nonzero and largest eigenvalues of the Laplacian matrix [8].

By defining the eigenvector corresponding to the largest eigenvalue as v_N, we can get the following formula

$$\lambda_N = \sum_{i \sim j} (v_N(i) - v_N(j))^2, \tag{2}$$

subject to $\sum_i v_N(i)^2 = 1$ [33]. This suggests that λ_N will stay unchanged if we remove the edge (i, j) which minimizes $|v_N(i) - v_N(j)|$ [15]. Actually, the same ideas work for the smallest nonzero eigenvalue λ_2 and associated eigenvector v_2. The analysis indicates that not every link contributes to the network synchronizability. We call these links as the redundant links for synchronization. In the next section, we will design algorithms to identify and delete the redundant links from networks.

3. The greedy algorithms to remove redundant links

We first denote the synchronizability of the original network as $R^{(0)}$. After removing l links, the synchronizability of the reduced network is denoted as $R^{(l)}$. Naturally, the most straightforward way to design the greedy algorithm is to remove the link which minimizes

$$\Delta R = |R^{(l)} - R^{(l-1)}|. \tag{3}$$

In practice, we calculate ΔR of each remaining link (by assuming this link is removed) and select the link with lowest ΔR to be actually removed from the network. With this method, one can effectively preserve the R value even when a considerable number of links are cut from the network. For convenience, this method is called Greedy Algorithm based on R (for short GAR).

We test the above algorithm in two kinds of modeled networks: (i) the Watts–Strogatz network (WS network) [34]: the model starts from a completely regular network with identical degree. Each link will be rewired with two randomly selected nodes with probability $q \in (0, 1)$. (ii) the Barabasi–Albert network (BA network) [35]: starting from m all to all connected nodes, at each time step a new node is added with m links. These m links connect to old nodes with probability $p_i = k_i / \sum_j k_j$, where k_i is the degree of the node i. Besides this standard BA model, we will consider later a variant of it