Physica A 400 (2014) 186–193

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Characterizing weak chaos in nonintegrable Hamiltonian systems: The fundamental role of stickiness and initial conditions

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HIGHLIGHTS

- The dynamics of weakly chaotic Hamiltonian systems is investigated via sticky effects.
- Sticky effects are detected using the finite time Lyapunov exponents (FTLEs).
- Sticky effects are quantified by the higher cummulants of the FTLEs distribution.
- The fundamental role of initial conditions is shown.

ARTICLE INFO

Article history:
Received 7 January 2014
Available online 20 January 2014

Keywords:
Finite time Lyapunov spectrum
Coupled maps
High-dimensional
Hamiltonian system
Stickiness

ABSTRACT

Weak chaos in high-dimensional conservative systems can be characterized through sticky effect induced by invariant structures on chaotic trajectories. Suitable quantities for this characterization are the higher cummulants of the finite time Lyapunov exponents (FTLEs) distribution. They gather the *whole* phase space relevant dynamics in *one* quantity and give information about ordered and random states. This is analyzed here for discrete Hamiltonian systems with local and global couplings. It is also shown that FTLEs plotted *versus* initial condition (IC) and the nonlinear parameter are essential to understand the fundamental role of ICs in the dynamics of weakly chaotic Hamiltonian systems.

1. Introduction

The phase space of nonlinear conservative Hamiltonian systems can present regions of regular, mixed and chaotic motion, depending on the nonlinear parameter. The regular region is characterized by the complete absence of chaotic trajectories while the chaotic region by the absence of regular trajectories, with exceptions of measure zero stable orbits. The mixed regions on the other hand, contains simultaneously the regular and chaotic motion (or quasi-regular) and everything becomes complicated [1,2]: the positions of regular structures, their size distribution, exit times, the shapes of the structures boundaries and the penetration inside them which is possible in higher dimensions. Weak chaos occurs in the mixed region where an approximately even competition between the regular and chaotic dynamics occurs [1]. In 2-dimensional conservative problems (4-dimensional phase space) the dynamics can be described by using the technique of Poincaré Surfaces of Section (PSS). However, for high-dimensional systems this technique is only partially useful due to the restriction of plots to 2- and 3-dimensions, which makes it almost impossible to construct adequate PSSs which allow

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0378-4371/$ – see front matter
http://dx.doi.org/10.1016/j.physa.2014.01.021

to describe the whole dynamics. Beside that, 3-dimensional plots of high-dimensional systems are fairly unsatisfactory projections from the whole system. A recent work proposes visualization of classical structures using phase space slices [3].

Another possibility is to use Lyapunov quantifiers which decide if a trajectory is chaotic or not. In the mixed region stickiness [4,5] affects the convergence of finite time Lyapunov exponents (FTLEs), but they contain essential information about the properties of the regular structures which live in the high-dimensional phase space. The properties of regular structures have been addressed on the same constant energy [6], by studying their dimensionality [7,8], and the almost invariant sets in continuous problems [9,10], to mention a few. The purpose of the present work is not to describe the invariant structures itself, but to quantify their effect on the dynamics. As explained below, it turns out that FTLEs are very suitable to do the job in the mixed regions. Any dynamical evolution of the system depends on the starting point in phase space and on the shape and number of regular structures of its surroundings. Since FTLEs are usually strongly dependent on the initial conditions and on the sticky motion, they can be used to quantify the amount of regular motion in high-dimensional phase spaces. It would be nice to check the stickiness influence on the smaller alignment index [11,12] which rapidly distinguish between ordered and chaotic trajectories in Hamiltonian flows.

A very appropriate way to quantify the sticky motion is by analyzing higher order cummulants of the FTLEs distribution. This was proposed [13] for the standard map and applied [14] to higher dimensions in conservative non-Hamiltonian systems. It was found [14] by this technique that for 2, 4, 10 and 20 phase space dimensions, conservative coupled standard maps with unidirectional local coupling can be characterized of being chaotic, quasi-regular or regular. In addition, for some values of the nonlinear parameter in the quasi-regular region, stickiness is shown to affect *all* unstable directions *simultaneously* and by the *same* amount, which is quantified by the cummulants mentioned above. This remarkable property was named *common* behavior [14] and its main property is that regular structures in phase space "attract" the chaotic trajectories by the same amount in all unstable directions. Once the chaotic trajectory is attracted, it remains sticked to the regular structure and then presents the clustering behavior observed [15] in conservative maps. But the essential new property from the common behavior is that the rate of attraction of the chaotic trajectory into the regular structure is equal in different unstable directions.

The question remains if the common motion is a general property found also in Hamiltonian and symplectic systems, or maybe it depends on the particular choice of the coupling? In order to answer this question and to understand better mixed phase spaces, in this work we extend our method to Hamiltonian systems with global and local couplings and compare the results. The existence of an additional constant of motion, besides the total energy, allows for a clear interpretation of results.

The paper is divided in the following way. In Section 2 the local and global coupled maps models used in this work are presented and in Section 3 we summary the main properties of the FTLEs distribution in order to detect stickiness. In Section 4 we present and discuss the results which are summarized in the conclusions in Section 5.

2. Lattices of coupled Hamiltonian maps

The model we investigate is conservative with an additional constant of motion. It describes N particles coupled on a unit circle, where the state of each particle is defined by its position $x^{(i)}$ and its conjugate momentum $p^{(i)}$. The lattice of N coupled maps is written as

$$\begin{cases} p_{t+1}^{(i)} = p_t^{(i)} + f(x_t) \quad \mod 1, \\ x_{t+1}^{(i)} = x_t^{(i)} + p_{t+1}^{(i)} \quad \mod 1, \end{cases} \tag{1}$$

where $f(x_t)$ can be local or global coupling as discussed next. The local coupling (LC) is defined accordingly to

$$f(x_t) = \frac{K}{2\pi}\left\{\sin[2\pi(x_t^{(i+1)} - x_t^{(i)})] - \sin[2\pi(x_t^{(i)} - x_t^{(i-1)})]\right\}, \tag{2}$$

where $i = 1, \ldots, N$. We considered periodic boundary conditions $p^{(N+1)} = p^{(1)}, x^{(N+1)} = x^{(1)}$. Some properties of this model were already investigated numerically in Refs. [16–18]. For the global coupling (GC) we have

$$f(x_t) = \frac{K}{2\pi\sqrt{N-1}} \sum_{j=1, j\neq i}^{N} \sin[2\pi(x_t^{(j)}) - 2\pi(x_t^{(i)})], \tag{3}$$

where $i = 1, \ldots, N$ [17]. K is simultaneously the nonlinear parameter and the coupling strength between distinct sites. For $K > 0$ the interaction term $f(x_t)$ between two particles i and j is attractive [19]. Both models have the total momentum $P_T = \sum_{j=1}^{N} p_t^j$ as a conservative quantity and they were extensively studied in Refs. [19,20], which characterized the system by the existence of an ordering process called cluster, which will be discussed later.