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# High frequency energy cascades in inviscid hydrodynamics

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ABSTRACT

With the aim of gaining insight into the notoriously difficult problem of energy and vorticity cascades in high dimensional incompressible flows, we take a simpler and very well understood low dimensional analog and approach it from a new perspective, using the Fourier transform. Specifically, we study, numerically and analytically, how kinetic energy moves from one scale to another in solutions of the hyperbolic or inviscid Burgers equation in one spatial dimension (1D). We restrict our attention to initial conditions which go to zero as $x \to \pm\infty$. The main result we report here is a Fourier analytic way of describing the cascade process. We find that the cascade proceeds by rapid growth of a crossover scale below which there is asymptotic power law decay of the magnitude of the Fourier transform.

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## 1. Introduction

It is sometimes useful to take a well understood problem and approach it from a new angle or viewpoint. The insight gained from the new perspective may help to allow further progress on more difficult problems. There are a number of very difficult open problems in the fields of hydrodynamics and mathematical fluid dynamics [1–5]. One of the most important unsolved problems concerns whether or not the three dimensional (3D) Euler equations for incompressible fluids admit finite time singularities, i.e. given an initially smooth velocity field, can the solution become singular or "blow up" in finite time? This problem is considered by experts to be very hard [3]. Slightly easier is the corresponding problem for viscous fluids. For Newtonian fluids, i.e. those that obey Fick's law, the 3D Navier–Stokes equations play the role of the Euler equations. The question of whether or not initially smooth solutions of the 3D Navier–Stokes equations for incompressible fluids can blow up in finite time is one of the main open problems in mathematical fluid dynamics. A key aspect of both these blow-up problems is that in 3D, the kinetic energy which is present at given spatial scales (i.e., wave vector scales) can, in principle, shift to the next finer spatial scale (i.e., to smaller wave vectors). Meanwhile, the corresponding time scales also get smaller, so that the process could, in principle, be repeated at ever finer scales and at ever faster rates, resulting in a finite time singularity. This iterative downscaling of the kinetic energy is often described as a "cascade" and is what makes turbulence possible. For comparison, in 2D such kinetic energy cascades to ever smaller scales cannot lead to finite time singularities. In fact, there is a reverse cascade, where the energy at small scales is transferred to larger scales. But in 3D there is still no known mechanism which prohibits finite time blow-ups, because there is nothing to prevent the kinetic energy from cascading to ever smaller spatial and temporal frequency scales. Our goal here is to gain a better understanding of energy

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cascades from lower to higher frequencies by studying a simpler system: the 1D inviscid (or hyperbolic) Burgers equation [6,7]. Here we use the 1D inviscid Burgers equation as a toy model to study energy cascades. The premise of this paper is that we can gain a deeper understanding of kinetic energy cascades in hydrodynamics by looking at the well understood inviscid Burgers equation.

In addition to the purely academic interest in this class of problems, there are some practical reasons for their study and we briefly mention a couple [8,9]. The circulation of blood through the large blood vessels such as the Aorta and the Vena Cava can sometimes become turbulent instead of laminar. Turbulent flow is only possible if kinetic energy is able to cascade down to the smallest scales. A second example is turbulence in natural gas pipelines. Problems of a slightly different kind also arise in situations triggered by fluid instabilities. In these phenomena, in general, the onset of the instability is characterized by low frequency modes. The system then develops energy cascades and evolves rapidly to a turbulent regime described by high frequency oscillations and fluctuations at all scales. An example is given by the Rayleigh–Taylor instability, in which a dense inviscid fluid is supported by another fluid of lower density in a gravitational field. This instability occurs also in magneto-hydrodynamics and is responsible for the phenomenon called "Spread F" in the F-region of the ionosphere. Another situation, common in the petroleum industry, is generated when a less viscous fluid is injected in a petroleum reservoir, for oil recovery, at high injection rates. This can result in bubbles, vortices and also fluctuations at many scales. In addition, in porous media (such as petroleum reservoirs), even laminar flows could theoretically induce vortices of many sizes. For instance, there is some interest in studying fluid flows through rough walls and porous materials that form a fractal structure with sharp edges and open irregular cavities.

In the sections that follow, we briefly review the nonlinear wave equations of hydrodynamics and the properties of the inviscid Burgers equation. We then turn our attention to the definition of the $L^p$ norms, as well as the known conservation laws for the inviscid Burgers equation. We then report our numerical and analytical results, the main one being that the Fourier transform of initially smooth rapidly decaying solutions lies in $L^p$ for all $p > 1$. Finally, we present our results, discussion, and conclusions.

## 2. Burgers equation and hydrodynamics

The 3D Euler equations for incompressible fluids for the velocity field $\mathbf{u}(\mathbf{x}, t)$ are the following:

$$\frac{\partial}{\partial t}\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = 0 \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0. \tag{2}$$

Here $p$ is the pressure. By taking the divergence of the first equation, one can invert the resulting elliptic equation to obtain $p = -\Delta^{-1}\mathrm{Tr}\,(\nabla\mathbf{u})^2$, where $\Delta^{-1}$ is the integral operator which is the inverse of the Laplacian operator $\Delta = \nabla^2$. The Navier–Stokes equations differ from the Euler equation only by an additional term for viscous dissipation. The viscosity can be taken to be unity, by a simple renormalization of the units. Then, the Navier–Stokes equations become

$$\frac{\partial}{\partial t}\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \Delta\mathbf{u}. \tag{3}$$

The 1D viscous Burgers equation bears a resemblance to the Navier–Stokes equation (both are parabolic):

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u = \Delta u. \tag{4}$$

Note that the viscous Burgers equation does not contain a pressure term. Indeed, Burgers equation represents a "fluid" which is perfectly compressible. The inviscid Burgers equation is simply

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u = 0, \tag{5}$$

i.e., there is no viscous dissipation. Like the Euler equations, the inviscid Burgers equation is hyperbolic and it is perhaps the simplest of the nonlinear hyperbolic wave equations.

At one time, it was conjectured that the viscous Burgers equation might be a good model of turbulence. It was hoped that the study of Burgers equation would lead to insights into the Navier–Stokes equations. However, a seminal result due independently to Hopf [10] and Cole [11] dashed those hopes. The Cole–Hopf transformation reduces the viscous Burgers equation to the heat equation, also known as the diffusion equation to physicists. The Cole–Hopf transformation is

$$u = \frac{-2}{\phi}\frac{\partial\phi}{\partial x} = -2\frac{\partial}{\partial x}\ln\phi \tag{6}$$

allowing a closed form solution of the initial value problem given $u(x, 0) = u_0(x)$:

$$u(x, t) = -2\tfrac{\partial}{\partial x}\ln\left[\tfrac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} \mathrm{e}^{(-\frac{(x-y)^2}{4t} - \frac{1}{2}\int_0^y u_0(s)ds)}\mathrm{d}y\right]. \tag{7}$$

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