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Quantum walks with memory on cycles

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HIGHLIGHTS

- Model of quantum walks with memory on a cyclic graph.
- Analysis of limiting probability distribution.
- Comparison of the models with the memoryless case.

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ABSTRACT

We study the model of quantum walks on cycles enriched by the addition of 1-step memory. We provide a formula for the probability distribution and the time-averaged limiting probability distribution of the introduced quantum walk. Using the obtained results, we discuss the properties of the introduced model and the difference in comparison to the memoryless model.

1. Introduction

During the last few years a considerable research effort has been made to develop new algorithms based on the rules of quantum mechanics. Among the methods used to achieve this goal, quantum walks, a quantum counterpart of random walks, provide one of the most promising and successful approaches.

Classical random walks can be applied to solve many computational problems. They are used, for example, to find spanning trees and shortest paths in graphs, to find the convex hull of a set of points or to provide a sampling-based volume estimation [1]. Today a huge research effort is devoted to applying random walks in different areas of science. Classical random walks find their application in a plethora of areas. This has motivated big interest in using a similar model for developing algorithms which could harness the possibilities offered by quantum machines.

Quantum walks are counterparts of classical random walks governed by the rules of quantum mechanics [2,3,1] and provide a promising method for developing new quantum algorithms. Among the applications of quantum walks one can point out: solving the element distinctness [4] and subset finding [5] problems, spatial search [6], triangle finding [7] and verifying matrix products [8]. The survey of quantum algorithms based on quantum walks is presented in Ref. [9].

The influence of memory on the behavior of quantum walks has been considered by Flitney et al. [10] and Brun et al. [11]. In Ref. [12] Mc Gettrick proposed a model of one-dimensional quantum walk on line with one-step memory and studied the limiting probability distribution for this model. This work was developed by Konno and Machida in Ref. [13]. More recently Rohde et al. considered a quantum walk with memory constructed using recycled coins and applied numerical experiments

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to study its properties. Moreover, an experimental proposal for implementing a quantum walk with memory using linear optics has also been considered in Ref. [14].

In this paper we introduce and study the model of quantum walks on cycle [15,16] enriched by the addition of 1-step memory [12]. Our main contribution is the calculation of the probability of finding the particle at each position after a given number of steps, and the limiting probability distribution for the introduced model. We also point out the differences between quantum walks on cycles with and without memory.

This paper is organized as follows. In Section 2 we introduce the model of a quantum walk on cycle with one-step memory. In Section 3 we analyze the introduced model using Fourier transform method and we discuss the behavior of the time-averaged limiting probability distribution of the discussed model. Finally, in Section 4 we summarize the obtained results and provide some concluding remarks.

2. The model

In the model discussed in Ref. [16] the space used by a quantum walk is composed of two parts—1-qubit coin and d-dimensional state space, *i.e.* $\mathcal{H} = \mathcal{H}_2 \otimes \mathcal{H}_d$. The shift operator in this case is defined as

$$S_0 = \sum_{c=0}^{1} \sum_{v=0}^{d-1} |c\rangle\langle c| \otimes |v + 2c - 1 \bmod d\rangle\langle v|. \tag{1}$$

Here we adopt this model and extend it with an additional register, referred to as memory register, which stores the history of a walk.

For a quantum walk with one step memory one needs a single qubit to store the history. In this case we use a Hilbert space of the form $\mathcal{H} = \mathcal{H}_2^{\mathrm{c}} \otimes \mathcal{H}_2^{\mathrm{m}} \otimes \mathcal{H}_d$, respectively for a coin, memory and a position.

As in the case of a memoryless walk, the first register is the coin register and the third register is used to encode the position of the particle. The second register stores the history of the walk. The history is encoded as direction from which the particle was moved in the previous move. If this register is in the state $|0\rangle$, the previous position of the particle was $n+1$. If this register is in the state $|1\rangle$, the previous position of the particle was $n-1$. The coin register indicates if the walk should continue in the previously chosen direction (transmission in state $|0\rangle$) or change the direction (reflection in state $|1\rangle$).

Taking into account the above, we define a shift operator for a quantum walk with a 1-step memory on cycle with d nodes as

$$S_1 = \sum_{n=0}^{d-1} \Big(|0\rangle\langle 0| \otimes |0, n-1 (\mathrm{mod}\ d)\rangle\langle 0, n| + |0\rangle\langle 0| \otimes |1, n+1 (\mathrm{mod}\ d)\rangle\langle 1, n|$$
$$+ |1\rangle\langle 1| \otimes |1, n+1 (\mathrm{mod}\ d)\rangle\langle 0, n| + |1\rangle\langle 1| \otimes |0, n-1 (\mathrm{mod}\ d)\rangle\langle 1, n| \Big), \tag{2}$$

or in a more consistent form as

$$S_1 = \sum_{n,m,c} |c\rangle\langle c| \otimes |h_{m,c}, n + 2h_{m,c} - 1 (\mathrm{mod}\ d)\rangle\langle m, n|, \tag{3}$$

where $h_{m,c} = m + c (\mathrm{mod}\ 2)$ represents a history dependence of the walk.

The walk operator is defined as toss-a-coin and make-a-move combination, *i.e.*

$$W_1 = S_1 (C \otimes \mathbb{1}_2 \otimes \mathbb{1}_d), \tag{4}$$

where C is a coin matrix, *e.g.* Hadamard matrix H

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{5}$$

or any matrix $C \in \mathrm{SU}(2)$.

The walk starts in some initial state $|\phi_0\rangle$. After each step the state is changed according to the formula

$$|\phi_n\rangle = W_1^n |\phi_0\rangle \tag{6}$$

or as a recursive relation

$$|\phi_{n+1}\rangle = W_1 |\phi_n\rangle. \tag{7}$$

The probability of finding a particle at position v after n steps is obtained after averaging over the coin and the memory registers

$$p(v, n) = \sum_{c,m} |\langle c, m, v|\phi_n\rangle|^2, \tag{8}$$

or, in other words, by tracing out over the memory and the coin subspaces

$$p(v, n) = \mathrm{tr}_{c,m} |\phi_n\rangle\langle\phi_n|, \tag{9}$$

where $\mathrm{tr}_{c,m}$ denotes the operation of tracing out with respect to the coin and the memory subspaces.