Physica A 393 (2014) 76–85

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Diffusion of relativistic gas mixtures in gravitational fields

Gilberto M. Kremer *

Departamento de Física, Universidade Federal do Paraná, Curitiba, Brazil

HIGHLIGHTS

- Fick's law in gravitational fields.
- Dependence of the diffusion coefficient on the gravitational field.
- Modified Marle model equation.

ARTICLE INFO

Article history:
Received 17 June 2013
Received in revised form 2 September 2013
Available online 19 September 2013

Keywords:
Relativistic gas mixtures
Schwarzschild metric
Fick law
Diffusion coefficient
Boltzmann equation
Modified Marle model equation

ABSTRACT

A mixture of relativistic gases of non-disparate rest masses in a Schwarzschild metric is studied on the basis of a relativistic Boltzmann equation in the presence of gravitational fields. A BGK-type model equation of the collision operator of the Boltzmann equation is used in order to compute the non-equilibrium distribution functions by the Chapman–Enskog method. The main focus of this work is to obtain Fick's law without the thermal-diffusion cross-effect. Fick's law has four contributions, two of them are the usual terms proportional to the gradients of concentration and pressure. The other two are of the same nature as those which appear in Fourier's law in the presence of gravitational fields and are related to an acceleration and a gravitational potential gradient, but unlike Fourier's law these last two terms are of non-relativistic order. Furthermore, it is shown that the coefficients of diffusion depend on the gravitational potential and become smaller than those in its absence.

1. Introduction

Research on the non-equilibrium properties of relativistic gases using the Boltzmann equation in gravitational fields is a subject few have explored in the literature. An important contribution was due to Bernstein [1], who obtained the constitutive equation for the non-equilibrium pressure of a relativistic gas and the corresponding transport coefficient of bulk viscosity in a Friedmann–Robertson–Walker metric.

Recently, a relativistic gas in a gravitational field was analyzed in order to determine the influence of the gravitational potential gradient on Fourier's law [2,3] and the influence of the gravitational potential on the transport coefficients [3].

According to Ref. [3] the heat flux in Fourier's law has three contributions: the usual temperature gradient and two relativistic terms. One of them – proposed by Eckart from a thermodynamic theory [4] – is connected to the inertia of energy and represents an isothermal heat flux when matter is accelerated. The other – suggested by Tolman [5,6] – requires that in the absence of an acceleration a state of equilibrium of a relativistic gas in a gravitational field is achieved only if the temperature gradient is counterbalanced by a gravitational potential gradient.

In this work we are interested only in analyzing Fick's law in the presence of gravitational fields without the determination of the thermal-diffusion cross-effect and of the heat flux with the corresponding diffusion-thermal cross-effect, which will be subject of a forthcoming paper. It differs from the work in [7], where diffusion in curved space–time

* Tel.: +55 41 33613091; fax: +55 41 33613418.
E-mail address: kremer@fisica.ufpr.br.

0378-4371/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.physa.2013.09.019

is analyzed by using the Fokker–Planck equation. As in [3], we use the Schwarzschild metric and analyze a relativistic gas mixture of constituents which have non-disparate rest masses.

Here we show that Fick's law in the presence of gravitational fields has four contributions. The usual contributions due to the concentration gradient and pressure gradient, as well as the two that appear in Fourier's law and are proportional to the acceleration and gravitational potential gradient. However, unlike Fourier's law these last two contributions are not of relativistic order. We have also obtained that the diffusion coefficients depend on the gravitational potential, becoming smaller than those in its absence.

The work is structured as follows. In Section 2 we introduce the Schwarzschild metric, the system of Boltzmann equations in the presence of a gravitational fields and the two first moments of the distribution functions with their corresponding balance equations. A BGK-type model of the Boltzmann equation is introduced in Section 3, which depends on a reference distribution function, determined from the assumption that the balance equations of the full Boltzmann equation and of the BGK-type model lead to the same production terms. The non-equilibrium distribution functions are calculated in Section 4 by using the Chapman–Enskog method. In Section 5 the constitutive equations for the diffusion fluxes – which correspond to Fick's law – are determined from the non-equilibrium distribution functions and the diffusion coefficients are obtained. In the last section the main conclusions of the work are stated. We close the work with two appendices. In the first it is shown how to calculate the production terms of the partial balance equations of the energy–momentum tensors, while in the second the components of the Christoffel symbols in a Schwarzschild metric are given.

2. Basic equations

We consider a relativistic gas mixture of r constituents in a Riemannian space with metric tensor $g_{\mu\nu}$, where the particles of constituent $a = 1, \ldots, r$ have rest mass m_a and are characterized by the space–time coordinates $(x^\mu) = (ct, \mathbf{x})$ and momenta $(p_a^\mu) = (p_a^0, \mathbf{p}_a)$. The length of the momentum four-vector is constant so that $g_{\mu\nu}p_a^\mu p_a^\nu = m_a^2c^2$, which implies that

$$p_a^0 = \frac{p_{a0} - g_{0i}p_a^i}{g_{00}}, \qquad p_{a0} = \sqrt{g_{00}m^2c^2 + \left(g_{0i}g_{0j} - g_{00}g_{ij}\right)p_a^ip_a^j}. \tag{1}$$

As in [3] we shall adopt the isotropic Schwarzschild metric

$$ds^2 = g_0(r)\left(dx^0\right)^2 - g_1(r)\delta_{ij}dx^idx^j, \qquad g_0(r) = \frac{\left(1 - \frac{GM}{2c^2r}\right)^2}{\left(1 + \frac{GM}{2c^2r}\right)^2}, \qquad g_1(r) = \left(1 + \frac{GM}{2c^2r}\right)^4, \tag{2}$$

where G denotes the gravitational constant and M the mass of the spherical source.

In terms of the isotropic Schwarzschild metric (2), Eqs. (1) reduce to

$$p_a^0 = \frac{p_{a0}}{g_0} \qquad p_{a0} = \sqrt{g_0}\sqrt{m_a^2c^2 + g_1|\mathbf{p}_a|^2}, \qquad \sqrt{-g} = \sqrt{g_0g_1^3} \tag{3}$$

where $g = \det(g_{\mu\nu})$.

The components of the four-velocity in the isotropic Schwarzschild metric are

$$U^\mu = \left(\frac{c}{\sqrt{g_0 - v^2/c^2}}, \frac{\mathbf{v}}{\sqrt{g_0 - v^2/c^2}}\right), \tag{4}$$

which in a co-moving frame ($\mathbf{v} = \mathbf{0}$) reduces to $U^\mu = \left(c/\sqrt{g_0}, \mathbf{0}\right)$.

A state of the relativistic mixture of r constituents in the phase space spanned by the space–time and momentum coordinates is described by the set of one-particle distribution functions $f(\mathbf{x}, \mathbf{p}_a, t) \equiv f_a$, $(a = 1, 2, \ldots, r)$ such that $f(\mathbf{x}, \mathbf{p}_a, t)d^3xd^3p_a$ at time t gives the number of particles of constituent a in the volume element d^3x about $\mathbf{x}$ and with momenta in the range d^3p_a about $\mathbf{p}_a$.

In the presence of gravitational fields the one-particle distribution function of constituent a satisfies the Boltzmann equation (see e.g. Ref. [8])

$$p_a^\mu\frac{\partial f_a}{\partial x^\mu} - \Gamma^i_{\mu\nu}p_a^\mu p_a^\nu\frac{\partial f_a}{\partial p_a^i} = \sum_{b=1}^{r}\int (f_a'f_b' - f_af_b)F_{ba}\sigma_{ab}\, d\Omega\sqrt{-g}\frac{d^3p_b}{p_{b0}}. \tag{5}$$

Here $\Gamma^i_{\mu\nu}$ are the Christoffel symbols, $F_{ba} = \sqrt{(p_a^\mu p_{b\mu})^2 - m_a^2m_b^2c^4}$ is the so-called invariant flux, while σ_{ab} and $d\Omega$ denote the invariant differential elastic cross-section and the element of solid angle that characterizes a binary collision between the particles of constituent a with those of constituent b, respectively. The binary collision is characterized by the momentum four-vectors of the particles of the two constituents p_a^α and p_b^α before collision and $p_a'^\alpha$ and $p_b'^\alpha$ after collision, which obey the energy–momentum conservation law $p_a^\alpha + p_b^\alpha = p_a'^\alpha + p_b'^\alpha$. Furthermore, the following abbreviations were introduced in (5):

$$f_a' \equiv f(\mathbf{x}, \mathbf{p}_a', t), \qquad f_b' \equiv f(\mathbf{x}, \mathbf{p}_b', t), \qquad f_a \equiv f(\mathbf{x}, \mathbf{p}_a, t), \qquad f_b \equiv f(\mathbf{x}, \mathbf{p}_b, t). \tag{6}$$