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Towards a definition of the Quantum Ergodic Hierarchy: Kolmogorov and Bernoulli systems

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HIGHLIGHTS

- We obtain a quantum version of the Ergodic Hierarchy: The Quantum Ergodic Hierarchy.
- The Casati–Prosen model and the kicked rotator illustrate the relevance of QEH.
- Weak limit is a necessary condition for the mixing level.
- Complexity of the QEH level conditions decreases with increasing chaos.
- QEH can characterize quantum chaos transitions which depend on a continuous parameter.

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ABSTRACT

In this paper we translate the two higher levels of the Ergodic Hierarchy [11], the Kolmogorov level and the Bernoulli level, to quantum language. Moreover, this paper can be considered as the second part of [3]. As in [3], we consider the formalism where the states are positive functionals on the algebra of observables and we use the properties of the Wigner transform [12]. We illustrate the physical relevance of the Quantum Ergodic Hierarchy with two emblematic examples of the literature: the Casati–Prosen model [13,14] and the kicked rotator [6–8].

1. Introduction

This paper is based on Ref. [1] which presents a three theories structure of classical chaos: Ergodic Hierarchy, Lyapunov exponents and Complexity. The first two theories are related to the Pesin theorem [2] and the last two ones to the Brudno theorem [1]. The Pesin theorem expresses the equivalence between the KS entropy and the exponential divergence of trajectories by the presence of positive Lyapunov exponents, and the positivity of these exponents is a necessary and sufficient condition for chaos. On the other hand, the Brudno theorem expresses the equivalence between the complexity of almost every point of the phase space and the KS entropy. The theoretical relation between these chaos indicators, KS entropy, complexity and Lyapunov exponents, is sketched in a "chaos pyramid" in Fig. 1. According to this structure Ergodic Hierarchy is one of the features of classical chaos. We have presented in our previous paper [3] a proposal to define the first two steps of a Quantum Ergodic Hierarchy, i.e. Quantum Ergodic and Quantum Mixing systems. Then the only purpose of this paper is to complete the Quantum Ergodic Hierarchy with two more steps: Quantum Kolmogorov and Quantum

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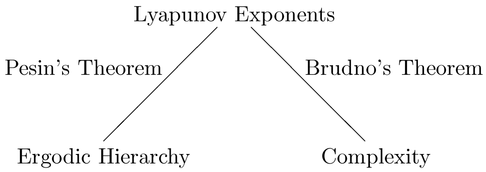

Fig. 1. The "chaos pyramid" is a diagram of the relationships between the three theories structures: Ergodic Hierarchy, Lyapunov exponents and Complexity through the Pesin and Brudno theorems.

Bernoulli. On the other hand the large majority of works on Quantum Chaos follows a different line. That is, in books [4–8] only "bra" and "ket" appear. Concepts like "observables" and "functionals" are not taken into account in a fundamental way in these books. Then a crucial definition of the mixing systems does not appear in these books, i.e. as the property of a quantum system with weak limit reaches equilibrium [3]. Essentially the way to introduce a Quantum Ergodic Hierarchy cannot be based on the study of quantum closed systems with just "bra" and "ket". Other concepts must be introduced as "observables" and "mixed states".

More precisely, following the Ballentine's book [9], where an axiomatic structure for Quantum Mechanics is sketched, the primitive concepts of Observable O are introduced. Then the states ρ are defined as a derived concept. They are the functional over the observable space. Then the "bra" and "ket" are simple vectors while the observables and states are matrices. Then somehow the usual treatment of quantum chaos in closed systems with the "bra" and "ket" is not enough, since this formalism does not consider open systems.

Moreover, as we explained in the introduction of Ref. [3], there are many ways to define quantum chaos. Among these ways, following part one, we will say that *"a quantum system is chaotic if its classical limit is chaotic"*, i.e. the Michael Berry's quantum chaos definition [10]. In this way we have defined quantum chaos in the two steps of the Quantum Ergodic Hierarchy: Ergodic and Mixing [3]. These steps are defined by their correspondent classical limit: they have a Cèsaro and a Weak limit respectively, that correspond to the limit of the Ergodic and Mixing systems of the classical ergodic hierarchy [11]. We will follow the same strategy with the other two levels.

Then, we have explained this limit in great detail. Precisely we have considered the Weyl–Wigner–Moyal transformation [12], the transformation symb that changes quantum operators and states into the corresponding classical symbols which become operators and states of classical analytic Mechanics when $\hbar \to 0$.

$$\operatorname{symb} \widehat{O} = O(q,p), \qquad \operatorname{symb} \widehat{\rho} = \rho(q,p), \quad \text{if } \hbar \to 0.$$

Then it can be shown that

$$\operatorname{Tr}(\widehat{\rho}\widehat{O}) = \int_{\Gamma} \rho(q,p) O(q,p) \mathrm{d}q \mathrm{d}p \tag{1}$$

where Γ is the phase space, and the last equation is valid even if $\hbar \neq 0$.

Thus, we define a classical mixing system as the one that satisfies the weak limit

$$\lim_{t\to\infty} \int_{\Gamma} \rho(q,p,t) O(q,p) \mathrm{d}q \mathrm{d}p = \int_{\Gamma} \rho_{*}(q,p) O(q,p) \mathrm{d}q \mathrm{d}p$$

where $\rho_{*}(q,p)$ is the weak equilibrium state of $\rho(q,p,t)$ and $O(q,p)$ belongs to the space of observables. Then from (1) the corresponding quantum chaos will satisfy the weak limit

$$\lim_{t\to\infty} \operatorname{Tr}(\widehat{\rho}(t)\widehat{O}) = \operatorname{Tr}(\widehat{\rho}_{*}\widehat{O})$$

namely the definition of quantum mixing chaos that we have given in paper [3] Section 6.3, Definition B.

This way to define the quantum version of the Ergodic Hierarchy, based in the classical one, is the one that we will also use in this paper (see Table 2).

As a consequence, following the ideas of Ref. [1] and the preceding paper [3], we will study the problem of quantum chaos hierarchy *directly from the quantum description of the chaotic classical limit*.

So as in paper [3], that we consider as the *first part* of this paper, we have defined the quantum chaos in the two first levels of the ergodic hierarchy (EH): ergodic and mixing. In this paper we will complete the work adding two more levels: Kolmogorov and Bernoulli. Then this paper can be considered as a *second part* of paper [3]. Nevertheless in this paper following the ideas of paper [11] we will first repeat the two initial levels using these new concepts and then adding the two final levels. Also, for the sake of conciseness, we will not repeat the following sections of paper [3]: Section 2 (Mathematical background), Section 3 (Decoherence in nonintegral systems), Section 4 (The classical statistical limit), and Section 5. (The classical limit.) These sections can be read in Ref. [3].

The paper is organized as follows. Section 2: We present the formalism, definitions and the Ergodic Hierarchy (EH) which we will use. Sections 3 and 4: we briefly review the ergodic and mixing systems already considered in Ref. [3]. Sections 5 and 6: We explain the Kolmogorov and Bernoulli cases in detail. Section 7: we give a physical relevance analyzing two emblematic examples of the literature in terms of the Quantum Ergodic Hierarchy: the Casati–Prosen model [13,14] and the kicked rotator [6–8]. Section 8: We consider the relevance of the subject and draw our conclusions.