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Born's formula from statistical mechanics of classical fields and theory of hitting times

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HIGHLIGHTS

- Classical field interacting with detector is modeled.
- Theory of first hitting times is applied.
- Born's rule is derived

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ABSTRACT

We consider Brownian motion in the space of fields and show that such a random field interacting with threshold type detectors produces clicks at random moments of time. The corresponding probability distribution can be approximately described by the same *mathematical formalism* as is used in quantum mechanics, theory of Hermitian operators in complex Hilbert space. The temporal structure of the "prequantum random field" which is the L_2-valued Wiener process plays the crucial role. Moments of detector's clicks are mathematically described as hitting times which are actively used in classical theory of stochastic processes. Born's formula appears as an approximate formula. In principle, the difference between the formula derived in this paper and the conventional Born's formula can be tested experimentally. In our model the presence of the random gain in detectors plays a crucial role. We also stress the role of the detection threshold which is not merely a technicality, but the fundamental element of the model.

1. Introduction

In this paper we present a novel application of statistical mechanics with the infinite-dimensional state space, the space of classical fields. We study the problem of interaction of a random field, e.g., the electromagnetic field, with a detector of the threshold type. We found that, for the natural random field, the Brownian motion in the space of fields, the L_2-valued Wiener process, statistics of clicks produced by such a detector can be *approximately* described by the same mathematical formula as is used in quantum mechanics (QM), Born's formula:

$$P_j = \mathrm{Tr}\rho\hat{C}_j, \tag{1}$$

where the operator ρ is Hermitian, positively defined and $\mathrm{Tr}\rho = 1$ and the operator $\hat{C}_j = |e_j\rangle\langle e_j|$ is a one dimensional projector. Although mathematically we obtained the same formula as in QM, the Hermitian operators in this formula have a special origin, namely, from the theory of classical random fields, and, hence, a new interpretation. The operator ρ is the covariance operator of the random field which is normalized by its trace and $\hat{C}_j$ gives the operator representation of the classical channel corresponding to the vector e_j.

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In QM the operator ρ represents the quantum state and the operator $\hat{C}_j$ represents the projective one dimensional measurement. The formula (1) with the aforementioned interpretation is known as *Born's rule.* This rule was postulated by Max Born [1] and it is used for the probabilistic interpretation of a quantum state; see introduction of the paper [2] for the historical analysis of invention of this rule, especially, connection with Schrödinger's interpretation of the squared wave function as the density of the electron charge. Although one cannot "derive an interpretation", the problem of "derivation of Born's rule" is actively discussed in quantum foundations.[1] In such studies one typically tries to obtain the mathematical formula (1) from some subquantum formalism and to establish the correspondence between operator quantities generated from the subquantum theory and the QM-quantities. To escape misinterpretation of our activity, we shall speak about derivation of Born's formula, by having in mind just the mathematical formula (1). We show that classical statistical mechanics of fields, theory of random fields [8–12], provides the possibility to represent detection probabilities in the operator form (1).

We state again that the formula (1) is derived as an approximate formula. In conventional QM this is the precise formula for calculation of detection probabilities.

The small parameter of the model determining the magnitude of deviation of the detection probability for a classical random field interacting with a threshold detector from (1) is the quantity

$$\epsilon \equiv \frac{\overline{\mathcal{E}}_{\text{pulse}}}{\mathcal{E}_d} \ll 1, \tag{2}$$

where $\mathcal{E}_d$ is the detection threshold having the physical dimension of energy and $\overline{\mathcal{E}}_{\text{pulse}}$ is the average energy of pulses emitted randomly by a source. Thus the theory is about random fields of low energy interacting with detectors with sufficiently high threshold. We derived the "precise formula" for the detection probability for the threshold type detectors. The basic idea behind this derivation is embedding of the problem of detection moments for the threshold detection into well developed theory of *hitting times* [13]. The latter is widely used in e.g. statistical radio-physics [14]. This theory gives us the probability distribution of detection times. The final precise formula, see (39), is quite complicated analytically. At the same time it is very simple from the viewpoint of numerical computation, summation of a series with quickly decreasing terms. In principle, this formula can be tested experimentally.[2]

This paper can be considered as development of *prequantum classical statistical field theory* (PCSFT) [15–24]. PCSFT provided representation of quantum averages and probabilities as averages and correlations with respect to ensembles of classical fields; including entangled systems (cf. Marshall [25], Hofer [5,6], de la Pena and Cetto [26], Casado et al. [27], Boyer [28], Cole [29], [30], Roychoudhuri [31,32], Adenier[33]).[3] So, QM and classical statistical mechanics became essentially closer than it was commonly believed. All quantum probabilistic quantities are represented in the classical probabilistic framework, cf. Manko et al. [35–43], Hess et al. [44,45], De Raedt et al. [46], Garola et al. [47]. PCSFT reproduced these quantities as averages of intensities of *continuous random fields.* However, quantum experimental statistics is based on *discrete events,* clicks of detectors. To come closer to experiment, PCSFT should be completed by measurement theory describing the transition from continuous random intensities to discrete random events of detection.

The first model of the threshold detection (TSD) of continuous random fields serving PCSFT was proposed in Ref. [2]. This paper was written on the physical level of rigorousness; mathematical conditions of applicability of the limiting procedure leading to Born's formula (1) were not discussed and the concrete class of stochastic processes for which such a procedure is applicable was not specified. Another crucial difference from the present paper is that in Ref. [2] from the very beginning a rough asymptotics of the detection probabilities was used; hence, directly Born's formula (1) was derived. Thus predictions of PCSFT/TSD were indistinguishable from the QM-predictions. The main advantage of the approach used in this paper that we were able to derive the exact formula for detection probabilities for classical random fields interacting with detectors

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1 Whether it is possible to derive this rule from some natural physical principles is still the subject of intensive debates on Born's work [1]. "The conclusion seems to be that no generally accepted derivation of Born's rule has been given to date, but this does not imply that such a derivation is impossible in principle". [3], cf. [4–7].

2 The precise formula for the threshold detection probability contains the probability distribution of the detector's gain. Determination of this distribution is a complicated problem depending essentially on types of threshold detectors. We shall discuss this problem in Sections 2.2 and 7.

3 The main problem of such approaches is to find the "right coupling" between the formalisms of classical field theory and quantum mechanics. If the latter is just an operational formalism, then its entities can be obtained as emergent from the fundamental prequantum model. Each of aforementioned approaches solved this problem in its own way. This situation is disturbing: there are a few mathematically different ways to obtain quantum theory from classical field theory. However, nowadays this situation is natural, since only experiment can determine the "rightness" of theory. And at the moment the prequantum effects are inapproachable. PCSFT can be distinguished from other prequantum approaches by two basic features of the prequantum→quantum correspondence: (a) the quantum density operator is simply the covariance operator of the prequantum field normalized by its trace; (b) the quantum observable is the image of the quadratic form of the prequantum field. In principle, by approaching the prequantum field component-wise one can test these hypotheses. We also remark that stochastic electrodynamics still preserves the classical particle picture for massive particles, e.g., for electron. PCSFT is a purely field model; "electron" as well as "photon" is a symbolic image of the corresponding prequantum field. Nelson's stochastic quantum mechanics [34] also preserves the classical particle image of, e.g., electron. In all these models, the background field plays the crucial role. In stochastic electrodynamics this is the zero point field of fundamental nature, so to say, vacuum fluctuations. It is not clear whether the PCSFT background field has to be interpreted as the fundamental "everywhere present" physical field. It might be that a form of the "noisy background interpretation" is sufficient. The problem has to be studied in more detail. Opposite to stochastic electrodynamics and Nelson's stochastic quantum mechanics, in Hofer's [5,6] electron density functional approach electron is treated as a field, i.e., this approach is closer to PCSFT.