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Statistical mechanics of phase–space curves

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ABSTRACT

We study the classical statistical mechanics of a phase–space curve. This unveils a mechanism that, via the associated entropic force, provides us with a simple realization of effects such as confinement, hard core, and asymptotic freedom. Additionally, we obtain negative specific heats, a distinctive feature of self-gravitating systems, and negative pressures, typical of dark energy.

1. Introduction

We will study here the classical statistical mechanics of arbitrary phase–space curves Γ and unveil some interesting effects, like confinement and hard-cores. Recall that by confinement one understands the physics phenomenon that impedes isolation of color charged particles (such as quarks), that cannot be isolated singularly. Therefore, they cannot be directly observed. In turn, asymptotic freedom is a property of some gauge theories that causes bonds between particles to become asymptotically weaker as distance decreases. Finally, in the case of a "hard core" repulsive model, each particle (usually molecules, atoms, or nucleons) consists of a hard core with an infinite repulsive potential.

Our curve-analysis will provide, in classical fashion, a simple entropic mechanism for these three phenomena. The so-called entropic force is a *phenomenological* one arising from some systems' statistical tendency to increase their entropy [1–5]. No appeal is made to any particular underlying microscopic interaction. The text-book example is the elasticity of a freely-jointed polymer molecule (see, for instance, Refs. [1,2] and references therein). However, Verlinde has argued that gravity can also be understood as an entropic force [3]. The same applies for the Coulomb force [6], etc. For instance, we have an exact solution for the static force between two black holes at the turning points in their binary motion [7] or investigations concerning the entanglement entropy of two black holes and an associated entanglement entropic force [8]. A causal path entropy (causal entropic forces) has been recently appealed to for links between intelligence and entropy [4].

Here we appeal to an extremely simple model to show that confinement can be shown to arise from entropic forces. Our model involves a quadratic Hamiltonian in phase–space.

Quadratic Hamiltonians are well known both in classical mechanics and in quantum mechanics. In particular, for them the correspondence between classical and quantum mechanics is exact. However, explicit formulas are not always trivial. Moreover, a good knowledge of quadratic Hamiltonians is useful in the study of more general quantum Hamiltonians (and

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their associated Schroedinger equations) for the semiclassical regime. Quadratic Hamiltonians are also important in partial differential equations, because they give non trivial examples of wave propagation phenomena. Quadratic Hamiltonians are also of utility because they help to understand properties of more complicated Hamiltonians used in quantum theory.

We wish here to appeal to quadratic Hamiltonians in a classical context in order to discern whether some interesting features are revealed concerning the entropic force along phase–space curves. We will see that the answer is in the affirmative.

2. Preliminaries

We consider a typical, harmonic oscillator-like Hamiltonian in thermal contact with a heat-bath at the inverse temperature β, that will be kept constant throughout:

$$H(p,q) = p^2 + q^2, \tag{1}$$

where p and q have the same dimensions (natural units, those of H, obviously; we wish to avoid dealing with a tensor g_{ij}).

The corresponding partition function is given by Refs. [9–11]

$$\begin{aligned} Z(\beta) &= \int_{-\infty}^{\infty} \mathrm{e}^{-\beta H(p,q)} \mathrm{d}p\mathrm{d}q \\ &= \pi \int_0^{\infty} \mathrm{e}^{-\beta U}\, \mathrm{d}U = \frac{\pi}{\beta}, \end{aligned} \tag{2}$$

where we employ the fact that the total microscopic energy is

$$U = p^2 + q^2 \tag{3}$$

and then we make the change of variable $p = \sqrt{U - q^2}$. Evaluating the resulting integral, first in the variable q and then in the variable U, we have for the mean value of the energy

$$\begin{aligned} \langle U(p,q)\rangle(\beta) &= \frac{1}{Z(\beta)} \int_{-\infty}^{\infty} H(p,q) \mathrm{e}^{-\beta H(p,q)}\, \mathrm{d}p\mathrm{d}q \\ &= \frac{\pi}{Z(\beta)} \int_0^{\infty} U \mathrm{e}^{-\beta U}\, \mathrm{d}U = \frac{\pi}{\beta^2 Z(\beta)}, \end{aligned} \tag{4}$$

and for the entropy

$$\begin{aligned} S(\beta) &= \frac{1}{Z(\beta)} \int_{-\infty}^{\infty} [\ln Z(\beta) + \beta H(p,q)] \mathrm{e}^{-\beta H(p,q)}\, \mathrm{d}p\mathrm{d}q \\ &= \frac{\pi}{Z(\beta)} \int_0^{\infty} \{\ln[Z(\beta)] + \beta U\} \mathrm{e}^{-\beta U}\, \mathrm{d}U = \frac{\pi}{\beta Z(\beta)} \{\ln[Z(\beta)] + 1\}. \end{aligned} \tag{5}$$

Note that the integrands appearing in (2), (4) and (5) are exact differentials.

3. Path entropy

Remember that we are in contact with a reservoir at the fixed inverse temperature β.

Path entropies (phase space curves) have been discussed recently in Refs. [4,5], for instance. We will be concerned here with a *related but not identical notion* and deal with a particle moving in phase space, focusing attention on its entropy evaluated as it moves along some phase space path Γ that starts at the origin and ends at some arbitrary point $(p_o(q_o), q_o)$. The path Γ is thus parameterized by the phase–space variable q. The usefulness of such a construct will become evident in the forthcoming sections. Also, as we will show below, some of the associated paths are adiabatic.

Accordingly, our purpose in this section is to define the thermodynamic variables of Section 2 *on these phase–space curves* Γ. It will be shown that this endeavor is useful. Remark that all our calculations here are of a microscopic character. No macrostates are to be dealt with at all! Thus, generalizing the exact differentials–integrands (2), (4) and (5) to curves Γ, we define the following.

- The partition function as a function of β and of a curve Γ

$$Z(\beta, \Gamma) = \pi \int_{\Gamma} \mathrm{e}^{-\beta U(p,q)}\, \mathrm{d}U(p,q). \tag{6}$$

- The mean energy as

$$\langle U(p,q)\rangle(\beta, \Gamma) = \frac{\pi}{Z(\beta,\Gamma)} \int_{\Gamma} U(p,q) \mathrm{e}^{-\beta U(p,q)}\, \mathrm{d}U(p,q). \tag{7}$$