Physica A 395 (2014) 458–465

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

A family of dissipative two-dimensional mappings: Chaotic, regular and steady state dynamics investigation

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HIGHLIGHTS

- In this work we consider a family of two dimensional mappings.
- We obtain the fixed points and an expression to find any periodic orbit with any period.
- For large nonlinearity we investigate the behaviour of the Lyapunov exponent.
- We characterise the steady state of the system for a long enough time.

ARTICLE INFO

Article history:
Received 16 October 2013
Available online 23 October 2013

Keywords:
Dissipative mapping
Chaotic dynamics
Stationary state

ABSTRACT

Some dynamical properties for a family of two dimensional mappings controlled by three parameters are considered in this paper. For null dissipation and depending on the other control parameters, diffusion in phase space is observed. A connection with the standard mapping is made to determine the range of control parameters leading to unlimited diffusion. In the dissipative case, for low nonlinearity, we obtain the fixed points and an expression to find any periodic orbit with any period, if it exists. For large nonlinearity we investigate the Lyapunov exponents and characterise the steady state of the system for a long time.

1. Introduction

The comprehension and characterisation of dynamical systems has been a challenging subject in recent years. The understanding of the dynamics leads to predictions of the steady state and an argument on the predictability of orbits for the long time dynamics, often searched for in many different systems ranging from several different areas [1–5].

Dynamical systems are mostly governed by a set of differential equations that in many cases are coupled to each other. Depending on the symmetries of the system and by using conserved quantities, a flow of solutions of the differential equations can be qualitatively (and many times quantitatively) transformed into an application described by nonlinear mappings [6].

Dissipation in mappings is present too [7–14]. The mappings are characterised by discrete time evolution and also by a set of control parameters. They can control either the nonlinearity as well as the dissipation itself. For conservative models, the choice of the control parameters can produce mixed phase spaces with chaotic seas surrounding periodic islands and

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http://dx.doi.org/10.1016/j.physa.2013.10.032

limited by invariant spanning curves. When dissipation is introduced in dynamics, the mixed structure of the conservative model is destroyed. Therefore it is common to also have attracting fixed points or sinks and chaotic attractors.

In this paper, we considered a family of two-dimensional nonlinear mappings described by the action and angle variables. The mappings are parametrised by an exponent γ and have a control parameter used to describe the dissipation, δ. The mapping also has a control parameter ϵ controlling the nonlinearity of the system. Our main goal in this paper is to investigate the influence of the dissipation in the behaviour of the Lyapunov exponents. Among this, we are also interested in characterising the periodic orbits. So far we shall find an expression to obtain any periodic orbit in the mapping, if they exist. The steady state showing the equilibrium is also discussed and analytical exponents are compared to previous results obtained in the literature using numerical simulations.

The paper is organised as follows. In Section 2 we discuss the map and find the analytical expression for the periodic orbits; Section 3 is devoted to discuss the chaotic dynamics and the steady state regime. Conclusions are drawn in Section 4.

2. The map, the phase space and characterisation of periodic orbits

We discuss in this section the mapping under consideration and the results obtained from either numerical simulations as well as the theoretical procedures. The description of the structure of the phase space and the periodic orbits determination is also made here. The mapping is a dissipative version [15,16] of a family of mappings considered previously [17] and is given by

$$T : \begin{cases} I_{n+1} = |\delta I_n - (1+\delta)\epsilon \sin(2\pi\theta_n)| \\ \theta_{n+1} = [\theta_n + I_{n+1}^{\gamma}] \text{mod } 1. \end{cases} \tag{1}$$

It is described by a set of two recursive equations for the dynamical variables, namely I and θ, corresponding respectively to the action and angle variables and containing three control parameters. The nonlinear term is given by a sine function whose nonlinearity is controlled by the parameter ϵ. $\gamma \neq 0$ in this paper is a free parameter that leads to the recovering of a set of different models in the literature. For $\gamma = 1$ the mapping (1) describes the so called bouncer model [18–23]. If $\gamma = -1$ it describes the Fermi–Ulam model [24,25] and for different values of γ than those previously described, many other applications appear [26–30]. The parameter δ is mainly related with the strength of the dissipation. For $\delta = 1$ the system is conservative and area preservation is observed in the phase space. However for $\delta < 1$ area contracting is dominating the phase space and attractors appear. Indeed the determinant of the Jacobian matrix is given by $\text{Det}\, J = \delta\, \text{sign}[\delta I_n - (1+\delta)\epsilon \sin(2\pi\theta_n)]$, where the function $\text{sign}(u) = 1$, if $u > 0$ and $\text{sign}(u) = -1$, if $u < 0$. For $\delta = 1$ and considering $\epsilon = 0$, the system is integrable and the phase space is filled only with regular dynamics, including periodic and quasi-periodic orbits. For $\epsilon \neq 0$ and considering $\gamma > 0$, the structure of the phase space can be presented as mixed while it is also possible to observe unlimited diffusion in the variable action under specific conditions. For $\delta < 1$, the mixed structure of the phase space is destroyed and attractors appear. For the combination of control parameters leading to diffusion in the action, the introduction of dissipation yields in a suppression of such a property producing in many cases the existence of a finite chaotic attractor in the phase space. This is so far a transition from unlimited to limited diffusion in the phase space.

Before presenting the phase space of the mapping, let us discuss first some properties for the particular case of $\gamma = 1$ and for the conservative dynamics, i.e. $\delta = 1$. Under these conditions, a comparison with the standard mapping [2] can be made easily. The equations describing the standard map are given by

$$T_{sm} : \begin{cases} I_{n+1} = I_n + K \sin(\phi_n) \\ \phi_{n+1} = [\phi_n + I_{n+1}] \text{mod } 2\pi. \end{cases} \tag{2}$$

It is well known that the standard mapping has a transition [2] from local to globally chaotic behaviour at $K_c = 0.9716\ldots$. For values of the parameter smaller than K_c, the phase space has a mixed structure containing both KAM islands, invariant spanning curves and chaotic seas. As soon as the parameter K becomes larger than K_c, no invariant spanning curves are observed anymore and unlimited diffusion in the action is possible.

Comparing the equations for mapping (1) and (2) we can see that they are correspondent to each other after the following transformations: (1) Multiplying both sides of both equations of mapping (1) by 2π; (2) Define the auxiliary variables $\tilde{\theta} = 2\pi\theta$ and $\tilde{I} = 2\pi I$; (3) Define a new variable as $\phi = \tilde{\theta} + \pi$. Then there is an effective control parameter $K_{\text{eff}} = 4\pi\epsilon$. The transition from local to globally chaotic behaviour in mapping (1) is given by $4\pi\epsilon > 0.9716\ldots$. Smaller values produce a mixed phase space with a chaotic sea of limited size due to the existence of invariant spanning curves. For larger values of the parameter, mixed structures can also be observed but unlimited diffusion in the action occurs for specific sets of initial conditions. This happens because the invariant spanning curves are destroyed.

Fig. 1 shows the phase space for the mapping considering the conservative dynamics for both cases with control parameter smaller ((a) $\epsilon = 0.065$) and larger ((b) $\epsilon = 0.085$) than that of the transition to global chaos. We can see from both figures a mixed structure but in Fig. 1(a) the existence of the invariant spanning curves prevent the chaotic sea to diffuse unlimited in the phase space. This is not the case for Fig. 1(b) when all the invariant spanning curves are destroyed and unlimited diffusion is observed. We see in both cases the structure of phase space repeats in action axis at each integer, $1, 2, \ldots$.