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Option volatility and the acceleration Lagrangian

Belal E. Baaquie [a,b], Yang Cao [a,*]

[a] Department of Physics, National University of Singapore, Singapore
[b] National University of Singapore and Risk Management Institute, Singapore

HIGHLIGHTS

- The model started from the classical solution of acceleration Lagrangian with certain boundary conditions.
- The option price was obtained by the conditional probability which is defined from the transition amplitude.
- The model provides a volatility formula as a function of future time, and fits the market USD/EUR option data very well.

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ABSTRACT

This paper develops a volatility formula for option on an asset from an acceleration Lagrangian model and the formula is calibrated with market data. The Black–Scholes model is a simpler case that has a velocity dependent Lagrangian.

The acceleration Lagrangian is defined, and the classical solution of the system in Euclidean time is solved by choosing proper boundary conditions. The conditional probability distribution of final position given the initial position is obtained from the transition amplitude. The volatility is the standard deviation of the conditional probability distribution. Using the conditional probability and the path integral method, the martingale condition is applied, and one of the parameters in the Lagrangian is fixed. The call option price is obtained using the conditional probability and the path integral method.

1. Introduction

Options are widely traded in exchanges and over-the-counter markets, and used for hedging, speculating and arbitrage [1]. There are two types of basic options — call and put options. The buyer of a call option has the right but not obligation to buy the underlying asset at a certain date for a certain price; the buyer of a put option has the right but not obligation to sell the underlying asset at a certain date for a certain price. The maturity of the contract is the expiration date; the pre-fixed price at maturity in the contract is called strike price.

The Black–Scholes model assumes that the stock price follows a log normal distribution, and derives the European call option price as a function of asset price, volatility and other parameters. The asset price, strike price, expiration date and interest rate are relatively easy to observe in the market. Given that these values are known, the option price depends on the volatility of the underlying asset. The volatility may be estimated from historical stock price as a constant. The volatility can also be implied by market option price obtained by inverting the Black–Scholes formula, which is known as implied volatility. The implied volatility with different expiration date or a different strike price generally yields different volatilities. In fact, option values in the market are quoted in terms of implied volatility rather than its price, and the implied volatility is a more useful measure of the option value than its price. Therefore, the pricing of options calls for estimations of volatility that matches the implied volatility.

* Corresponding author.
E-mail addresses: phybeb@nus.edu.sg (B.E. Baaquie), phycy.nus@gmail.com, yangcao@nus.edu.sg (Y. Cao).

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Baaquie [2] introduced the velocity Lagrangian for the Black–Scholes model in 1998. He obtained the Black–Scholes formula from the path integral. Besides, the acceleration Lagrangian have achieved great success in quantum finance interest rate models [3]. The quantum finance interest rate models show that the acceleration Lagrangian provides an excellent fit for the forward interest rate correlation in future time. In addition, the acceleration Lagrangian also gives a quite good fit for the equity correlation in calendar time, as discussed in Ref. [4]. Expiration time of an option is in the future of calendar time, and thus it is expected that acceleration Lagrangian may lead to an appropriate formula for the volatility as a function of expiration time. Moreover, the acceleration Lagrangian in Euclidean space is also a very interesting extension to physics with many applications [5–7].

Similar to the Black–Scholes model, the acceleration model assumes the following:

- the market is efficient;
- the investors are risk neutral;
- one can buy and sell the stock at any amount;
- the transactions do not incur any fees or costs;
- the underlying asset does not pay a dividend.

The risk neutral assumption leads to the martingale condition, namely that the present value of a European option is equal to the expected future payoff discounted by the risk-free interest rate.

Both velocity and acceleration Lagrangian models are solved by the same procedure. The velocity Lagrangian model is the path integral solution of the famous Black–Scholes model. However, the volatility of the Black–Scholes model is a constant, which is inconsistent with market data. The acceleration Lagrangian model improves the results, and generates a maturity dependent formula of volatility. Changing calendar time t to market time $z = t^{\eta}$, the volatility formula excellently fits the currency option market data, with an accuracy above 96.5%.

This paper is organized as follows: Section 1 introduces the key concepts in pricing options by classical solution and path integral method. Section 2 derives the volatility, martingale condition and option price for the acceleration Lagrangian. In Section 3, the infinite time limits of volatility and martingale condition are discussed. The data used for calibration is introduced in Section 4, and calibration results are shown in Section 5. Section 6 is the conclusion.

The details of the solution are shown in the Appendices. Appendix A gives the derivation of the Black–Scholes model by this method, which provides a clear guideline for the acceleration model. Appendix B shows the detailed expression of the classical action, volatility and infinite time limit of the acceleration Lagrangian. Appendix C derives the classical action with another boundary conditions, and the infinite time limit gives the ground state of acceleration Lagrangian.

1.1. *Option pricing by conditional probability and path integral*

Option price is the expectation value of payoff discounted to its present value. Payoff is the value realized by the holder at the end of the option life, denoted by $\mathcal{P}(x_f)$. The rate of return x is similar to position, and follows a normal distribution in option pricing. The conditional probability distribution $P(x_f|x_i)$ of final position x_f, given the initial position x_i is crucial for calculating the option price. Since the initial position x_i in option pricing is known, the summation of the possibility that x_f will happen is one. Thus $P(x_f|x_i)$ should satisfy the following normalization condition:

$$\int_{-\infty}^{+\infty} dx_f P(x_f|x_i) = 1. \tag{1}$$

Given $P(x_f|x_i)$, the expectation of the payoff $\mathcal{P}(x_f)$ is

$$E[\mathcal{P}(x_f)] = \int_{-\infty}^{+\infty} dx_f P(x_f|x_i)\mathcal{P}(x_f). \tag{2}$$

The volatility of a stock is defined as the standard deviation of rate of return x_f; hence the variance of conditional probability distribution $P(x_f|x_i)$ is equivalent to the square of volatility.

Additionally, option pricing models often assume that the market is efficient and risk neutral. As a result, the option price obeys the martingale condition, which means that the current value of the stock is equal to its future discounted value. Let S be the initial price, with $S = e^{x_i}$; the martingale condition yields, for the remaining time τ,

$$e^{x_i} = e^{-r\tau}E[e^{x_f}] = e^{-r\tau}\int_{-\infty}^{+\infty} dx_f P(x_f|x_i)e^{x_f}, \tag{3}$$

where the $e^{-r\tau}$ is the discounting factor and r is the interest rate.

The payoff of call option is $[e^{x_f} - X]_+$, where X is the strike price and τ is the remaining time. Consequently, the call option price $C(X, \tau)$ is given by

$$\begin{aligned} C(X,\tau) &= e^{-r\tau}E\Big[(e^{x_f} - X)_+\Big] \\ &= e^{-r\tau}\int_{-\infty}^{+\infty} dx_f P(x_f|x_i)\Big(e^{x_f} - X\Big)_+. \end{aligned} \tag{4}$$