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A small-world network derived from the deterministic uniform recursive tree

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ABSTRACT

As the deterministic version of the uniform recursive tree (URT), the deterministic uniform recursive tree (DURT) has been intensively studied by Zhang *et al.* (2008) [21]. They gave several important properties of DURT, including its topological characteristics and spectral properties. Although DURT shows a logarithmic scaling with the size of the network, DURT is not a small-world network since its clustering coefficient is zero. In this paper, we propose a new deterministic small-world network by adding some edges with a simple rule in each DURT iteration, and then give the analytic solutions to several topological characteristics of the model proposed.

1. Introduction

Small-world properties can be found in many real-world networks [1–5], such as road maps, food chains, electric power grids, metabolite processing networks, networks of brain neurons, voter networks, telephone call graphs, and social influence networks. The first property of small-world networks is that their average path length (APL) grows proportionally to the logarithm of the number of nodes in the network. The second typical property of small-world networks is that they have a high clustering coefficient [6] that makes them tend to contain cliques. Several other properties are often associated with small-world networks. For example, most of the small-world networks contain a large number of hubs (nodes with a high degree) that mediate the short path lengths between other edges. In addition, the average node degree of small-world networks is generally small.

In the past dozen years, a number of random models have been proposed to describe real-life small-world networks. The pioneer small-world network named WS model was introduced by Watts and Strogatz [6] in 1998. This work started a great deal of research on the properties of small-world networks. Another thoroughly-studied small-world network named NW model was proposed by Newman and Watts [7,8]. The WS model is generated by rewiring each edge of the regular ring lattice with a certain probability, while the NW model is generated by adding edges between each pair of unlinked nodes in the regular ring lattice with a certain probability. Furthermore, Kasturirangan [9] presented an alternative version called the $R + T$ network to the WS model, and it is actually a regular network coupled with a tree structure. Subsequently, Kleinberg proposed a generalization of the WS model based on a two-dimensional lattice [10]. In addition, Ozik et al. introduced a simple evolution model to grow small-world networks with geographical attachment preference [11].

The above small-world models are all random since they link new nodes based on a probabilistic rule to the nodes that already exist in the system. Although the randomness is one of the major features to mimic the generation process of real-life networks, it cannot provide us with a vivid understanding of how networks are formed. On the other hand, if a real-life

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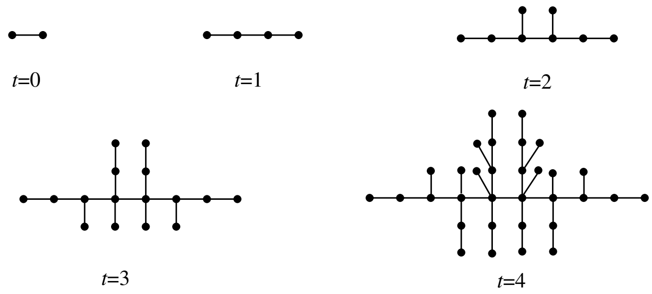

Fig. 1. The first five iterations of the growth process of DURT [21].

network (e.g., an electronic circuit) has fixed interconnections, the probabilistic technique is no longer suitable to model its generation process. Based on the above considerations, many researchers turn their hand to constructing scale-free or small-world networks in deterministic manners. The main advantage of deterministic networks lies in that their topological properties can be computed analytically. Comellas et al. [12] proposed the first deterministic small-world network based on graph-theoretic methods in 2000. Recently, Boettcher et al. [13] constructed a special type of deterministic small-world networks named "Hanoi Networks" based on the famous tower of Hanoi puzzle. The above two models are static models, for the number of nodes is fixed during the generation process. To obtain the growth model of small-world networks, Zhang et al. presented a deterministic small-world network created by edge iterations [14]. In fact, there are also some other mechanisms that can be used to generate deterministic small-world networks. In 2002, Comellas and Sampels presented two deterministic small-world models with constant and variable degree distributions respectively [15]. In 2004, based on prime numbers decomposition, Corso [16] constructed a deterministic small-world model in which the natural numbers are the vertices. In 2005, inspired by Corso's work, Chandra and Dasgupta constructed a small-world of prime numbers [17]. In 2006, Xiao and Parhami proposed a deterministic small-world model based on Cayley graphs [18].

The uniform recursive tree (URT) [19] is another important random model besides the Erdös–Rényi random graph. The URT can be generated in the following simple way: at each iteration, a new node is attached to an existing node that is randomly chosen. In 2002, a deterministic version [20] of the URT has been proposed to mimic real-life systems whose number of nodes increases exponentially with time. This kind of deterministic models have drawn much attention from the scientific communities and have turned out to be a useful tool. In 2008, Zhang et al. [21] have offered a detailed analysis of the deterministic uniform recursive tree (DURT) from the viewpoint of complex networks. They first derived topological characteristics of DURT, such as degree distribution, average path length, betweenness distribution, and degree correlations, and then calculated the eigenvalues and eigenvectors of the Laplacian matrix. The DURT's average path length shows a logarithmic scaling with the size of the network, however its clustering coefficient is zero. The purpose of this paper is to derive a small-world network from the DURT by adding some edges in each iteration with a simple rule, and thus we can get a high clustering coefficient.

The remainder of this paper is organized as follows. Section 2 reviews the deterministic uniform recursive tree and its main topological properties. Section 3 first proposes the iteration algorithm to generate the deterministic small-world network proposed, and then gives analytical results of the main network properties, including degree distribution, clustering coefficient and diameter. Section 4 concludes the whole paper.

2. Deterministic uniform recursive tree

The deterministic uniform recursive tree (DURT) is one of the simplest models that can be constructed by a simple iterative algorithm [20]. Let us denote the DURT obtained after t iterations as U_t that has N_t nodes and E_t edges, where $t = 0, 1, 2, \ldots, T-1$, and T is the total number of iterations, then the DURT generation process can be illustrated as follows:

Step 0: Initialization. Set $t = 0$, U_0 contains an edge that connects two nodes, and thus $N_0 = 2$ and $E_0 = 1$.

Step 1: Generation of U_{t+1} from U_t. For each node in U_t, a new node is linked to it. Thus we have $N_{t+1} = 2N_t$ and $E_{t+1} = E_t + N_t$.

Step 2: If $t < T-1$, set $t = t+1$ and go to Step 1. Otherwise, the algorithm is terminated.

The above iterative process is repeated for $T-1$ times, and then we can obtain a deterministic tree with $N_t = 2^{t+1}$ nodes and $E_t = 2^{t+1} - 1$ edges. Fig. 1 shows the network obtained after the first five iterations.

Zhang et al. have derived several important topological characteristics of DURT [21]. First, the cumulative degree distribution $P_{cum}(k) = 2^{-k+1}$ decays exponentially with k, which shows that the deterministic uniform recursive tree is an exponential network and has a similar form of degree distribution to the random uniform recursive tree [19]. Second, the average path length $d_t = (t \cdot 2^{t+1} + 1)/(2^{t+1} - 1)$, when $t \to \infty$, $d_t \to \ln N_t / \ln 2 - 1$. Thus, the APL shows a logarithmic scaling with the size of the network, indicating a similar small-world behavior as the URT [19] and the Watts–Strogatz (WS) model. Third, the betweenness distribution exhibits a power law behavior with exponent 2; the same scaling has been also obtained for the URT [19]. Fourth, the degree–degree correlation $k_{nn}(k)$ is approximately a linear function of k, which shows