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Option pricing, stochastic volatility, singular dynamics and constrained path integrals

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HIGHLIGHTS

- We study stochastic volatility models for extremely correlated cases.
- For each model we determine the underlying singular classical dynamics.
- We apply Dirac's method to singular Hamiltonian systems.
- We compute propagators in Euclidean time, for several different stochastic volatility models.

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ABSTRACT

Stochastic volatility models have been widely studied and used in the financial world. The Heston model (Heston, 1993) [7] is one of the best known models to deal with this issue. These stochastic volatility models are characterized by the fact that they explicitly depend on a correlation parameter ρ which relates the two Brownian motions that drive the stochastic dynamics associated to the volatility and the underlying asset. Solutions to the Heston model in the context of option pricing, using a path integral approach, are found in Lemmens et al. (2008) [21] while in Baaquie (2007,1997) [12,13] propagators for different stochastic volatility models are constructed. In all previous cases, the propagator is not defined for extreme cases $\rho = \pm 1$. It is therefore necessary to obtain a solution for these extreme cases and also to understand the origin of the divergence of the propagator. In this paper we study in detail a general class of stochastic volatility models for extreme values $\rho = \pm 1$ and show that in these two cases, the associated classical dynamics corresponds to a system with second class constraints, which must be dealt with using Dirac's method for constrained systems (Dirac, 1958,1967) [22,23] in order to properly obtain the propagator in the form of a Euclidean Hamiltonian path integral (Henneaux and Teitelboim, 1992) [25]. After integrating over momenta, one gets an Euclidean Lagrangian path integral without constraints, which in the case of the Heston model corresponds to a path integral of a repulsive radial harmonic oscillator. In all the cases studied, the price of the underlying asset is completely determined by one of the second class constraints in terms of volatility and plays no active role in the path integral.

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1. Introduction

The Black–Scholes model [1,2], one of the cornerstones of current financial theory, assumes by default that market volatility is constant. But the analysis of the actual economically-financial data implies, as a matter of fact, that volatility varies en time [2]. As Fisher Black himself remarked:

"Suppose we use the standard deviation of possible future returns on a stock as a measure of its volatility. Is it reasonable to take that volatility as a constant over time? I think not".

To address this problem in the context of the Black–Scholes standard model, the concept of smile has been developed [2–5]. In this approach, the volatility smile as a function of the underlying asset price is determined from the empirical data. Other more sophisticated models that try to capture the variation in volatility are stochastic volatility models [2,6]. Here it is assumed that the market volatility behaves as a random variable determined by a Brownian motion. This movement is different from a second Brownian motion that dictates the dynamics of the underlying asset in the Black–Scholes model. The correlation between both movements is parameterized by the correlation coefficient ρ. This fact, together with the hypothesis of no arbitrage and a self financing portfolio, imply that the option price satisfies a partial differential equation in two spatial dimensions (corresponding to the option price and the value of the volatility) and one time variable. The equation has a potential term similar to that of a quantum particle in the presence of an external electromagnetic field. Different stochastic models can be constructed depending on the shape of the potential term. The best known ones are the Heston model, the Hull and White model and the model of Ornstein–Uhlenbeck [2,7,8]. Other variations of these stochastic volatility models include incorporating a "jump diffusion" term, which gives rise to integro-differential equations [9] for the price of the option.

Furthermore, in recent years path integrals techniques have been increasingly applied to obtain solutions of the Black–Scholes equation [10–16] while numerical techniques are presented in Refs. [17–20]. In the context of stochastic volatility models, in Ref. [21] the propagator has been calculated for the Heston model via path integrals, obtaining closed form solutions for the price of the option. This result depends on the value of the correlation coefficient ρ and the proposed solution is indeterminate for extreme cases where $\rho = \pm 1$. The same behaviour appears in Refs. [12,13] where propagators for different stochastic volatility models are constructed. So, it is then interesting to study in detail what happens to the propagator when the correlation coefficient takes its extreme values $\rho = \pm 1$.

This article shows that, when looking at stochastic volatility models as Euclidean quantum mechanical systems, the classical mechanics underlying the bidimensional Schrödinger equation at $\rho = \pm 1$, is a system with constraints, reminiscent of an optimal control problem. Due to the presence of links, it is necessary to resort to the Dirac method [22–24] for the correct description of this constrained system, both in the classical and the quantum levels. Applying Dirac method shows that the links are second class and the propagator must be calculated in terms of a Hamiltonian path integral with second class constraints [25].

2. Stochastic volatility models

Stochastic volatility models are used to evaluate options prices and generalizes the Black–Scholes model to the non constant volatility case. Between the different plethora of models, the best know are the Heston model [7], the CEV model [26,27], the SBR volatility model [28], the GARCH model [29], the 3/2 model, the Hull and White model [8] and the Chen model [30]. Some stochastic volatility models are even capable to capture some important statistical properties of real markets, called stylized facts, such as the autocorrelation and leverage effect [31–34].

To account this stylized facts for the real financial data, empirical analysis implies that $|\rho|$ must be of the order of 0.5 [31,32]. Although, from a statistical point of view, the extreme case $\rho = \pm 1$ is not realized in the real world, these values can be satisfied for "outliers" events in the sense of Ref. [35], and from a structural perspective, is necessary understand the behaviour of the stochastic volatility models for the full range of its parameters values.

In order make contact with results for $\rho \neq \pm 1$ studied in literature [21,12,13], we consider first a wide class of stochastic models that are related to the Heston model, but our methods and analysis can be applied to an arbitrary stochastic volatility model. So, we consider a class of stochastic volatility models that are characterized by the following stochastic differential equations associated [3,2] to the underlying price $S(t)$ and the volatility $v(t)$ respectively

$$\mathrm{d}S = \mu(S,t)\mathrm{d}t + \sqrt{v}S\mathrm{d}W_1 \tag{1}$$

$$\mathrm{d}v = \alpha(S,v,t)\mathrm{d}t + \sigma\beta(S,v,t)\sqrt{v}\mathrm{d}W_2 \tag{2}$$

where it is assumed that the Brownian motions $\mathrm{d}W_1$ and $\mathrm{d}W_2$ have a correlation factor given by

$$\langle \mathrm{d}W_1 \ \mathrm{d}W_2 \rangle = \rho. \tag{3}$$

These equations, together with the assumption of no-arbitrage and using a self financing portfolio constructed from the underlying asset and options, imply [3,2] the following equation for the option price $U(S,v,t)$

$$\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma v\beta S\frac{\partial^2 U}{\partial S\partial v} + \frac{1}{2}\sigma^2\beta^2 v\frac{\partial^2 U}{\partial v^2} + r\left(S\frac{\partial U}{\partial S} - U\right) + (\alpha - \phi\beta\sqrt{v})\frac{\partial U}{\partial v} = 0 \tag{4}$$