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Inverted rank distributions: Macroscopic statistics, universality classes, and critical exponents

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HIGHLIGHTS

- Inverted rank distributions are macroscopically studied and socioeconophysically classified.
- The macroscopic classification is universal, and is governed by two critical exponents.
- Two classes, communism and socialism, manifest a "disordered phase".
- Two classes, feudalism and absolute monarchy, manifest an "ordered phase".
- A fifth class, criticality, is a "phase transition" between order and disorder.

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ABSTRACT

An inverted rank distribution is an infinite sequence of positive sizes ordered in a monotone increasing fashion. Interlacing together Lorenzian and oligarchic asymptotic analyses, we establish a macroscopic classification of inverted rank distributions into five "socioeconomic" universality classes: communism, socialism, criticality, feudalism, and absolute monarchy. We further establish that: (i) communism and socialism are analogous to a "disordered phase", feudalism and absolute monarchy are analogous to an "ordered phase", and criticality is the "phase transition" between order and disorder; (ii) the universality classes are characterized by two critical exponents, one governing the ordered phase, and the other governing the disordered phase; (iii) communism, criticality, and absolute monarchy are characterized by sharp exponent values, and are inherently deterministic; (iv) socialism is characterized by a continuous exponent range, is inherently stochastic, and is universally governed by continuous power-law statistics; (v) feudalism is characterized by a continuous exponent range, is inherently stochastic, and is universally governed by discrete exponential statistics. The results presented in this paper yield a universal macroscopic socioeconophysical perspective of inverted rank distributions.

1. Introduction

"*Rank distribution*" is the statistical term for a set of positive numerical values, often describing sizes, which are ordered monotonically. Examples of rank distributions include: (i) city sizes in a given country [1–3]; (ii) word frequencies in a given text (the size of a word being its number of occurrences in the text) [4–6]; (iii) firm sizes and firm bankruptcies [7–9]; (iv) measures of nodes and links in networks [10–12]; and (v) the Forbes lists of richest individuals [13]. Although the study

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of rank distributions has a century-long history, a most vibrant and exciting exploration of rank distributions has flourished recently [14–20].

Commonly, rank distributions are ordered *from large to small*. Hence, a given rank distribution is represented by a finite sequence of monotone decreasing sizes

$$S(1) \geq S(2) \geq \cdots \geq S(n), \tag{1}$$

where: $S(1)$ is the size of the largest element (ranked #1), $S(2)$ is the size of the second largest element (ranked #2), and $S(n)$ is the size of the smallest element (ranked #n). Evidently, the representation of Eq. (1) considers a *finite* number of underlying elements (n). Often however, it is natural to consider an *infinite* number of elements. For example, if we rank the stars in the known universe by their mass then, for all practical purposes, the natural setting is infinite rather than finite. The infinite counterpart of Eq. (1) is an infinite sequence of monotone decreasing sizes

$$S(1) \geq S(2) \geq S(3) \geq \cdots \tag{2}$$

where: $S(1)$ is the size of the largest element (ranked #1), $S(2)$ is the size of the second largest element (ranked #2), $S(3)$ is the size of the third largest element (#3), etc. A detailed macroscopic statistical analysis of the infinite rank distribution of Eq. (2) – yielding macroscopic universality classes and phase transitions – was presented in Ref. [21].

The transition from the finite setting of Eq. (1) to the infinite setting of Eq. (2) implicitly maintains the commonplace monotone decreasing ordering of sizes. In a finite setting ranking increasingly or decreasingly is effectively equivalent. Does this monotonicity equivalence hold when transcending from a finite to an infinite setting? In this paper we show that the answer is a categorical "*no*". To that end we reverse the order of ranking – ordering now increasingly rather than decreasingly – and consider *inverted rank distributions* represented by an infinite sequence of monotone increasing sizes

$$S(1) \leq S(2) \leq S(3) \leq \cdots \tag{3}$$

where: $S(1)$ is the size of the smallest element (ranked #1), $S(2)$ is the size of the second smallest element (ranked #2), $S(3)$ is the size of the third smallest element (#3), etc.

In this paper we present a detailed macroscopic statistical analysis of the inverted rank distribution of Eq. (3), which interlaces together Lorenzian and oligarchic asymptotic analyses. These analytic methods were successfully applied in Refs. [21–25]. The effectiveness of the Lorenzian and oligarchic analyses proves successful yet again, as applying these methods we establish the following socioeconophysical macroscopic results:

- The inverted rank distribution of Eq. (3) belongs to one of five *universality classes*: communism, socialism, criticality, feudalism, and absolute monarchy.
- The universality classes of communism and socialism are analogous to a *disordered phase*, the universality classes of feudalism and absolute monarchy are analogous to an *ordered phase*, and the critical universality class is the *phase transition*—the very boundary between disorder and order.
- The five universality classes are characterized by two *critical exponents*—one governing the disordered phase, and the other governing the ordered phase.
- The universality classes of communism, criticality, and absolute monarchy are characterized by sharp values of the critical exponents, and are inherently deterministic.
- The universality class of socialism is characterized by a continuous critical-exponent range, is inherently stochastic, and is universally governed by the inverse-Pareto distribution—manifesting *continuous power-law statistics*.
- The universality class of feudalism is characterized by a continuous critical-exponent range, is inherently stochastic, and is universally governed by the geometric distribution—manifesting *discrete exponential statistics*.

The macroscopic picture emerging from these results is profoundly and markedly different from the counterpart macroscopic picture attained in the context of the infinite rank distribution of Eq. (2) [21]. Thus, as proclaimed above, the aforementioned "monotonicity equivalence" indeed breaks down when transcending from a finite to an infinite setting.

The paper is organized as follows. Section 2 tersely reviews the notions of Lorenz curves and regular variation, which are required for the macroscopic statistical analysis. The Lorenzian and oligarchic asymptotic analyses are carried out, respectively, in Sections 3 and 4. Interlacing these asymptotic analyses we present, in Section 5, a macroscopic classification of the inverted rank distribution of Eq. (3) into the five socioeconomic universality classes noted above. Section 6 demonstrates the application of the theory via examples.

2. Preliminaries

In this section we tersely review two notions that will facilitate the analysis of inverted rank distributions: the statistical notion of *Lorenz curves*, and the mathematical notion of *regular variation*.