Physica A 393 (2014) 527–534

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

# Optimal structure of tree-like branching networks for fluid flow

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## HIGHLIGHTS

- We derive some dimensionless expressions of effective flow resistance.
- The effects of structural parameters on the flow resistance are studied.
- We give optimal design schemes when the flow resistance of the whole network is minimum.

## ARTICLE INFO

*Article history:*
Received 22 February 2011
Received in revised form 19 February 2013
Available online 20 September 2013

*Keywords:*
Flow resistance
Fractal tree-like branching network
Optimal structure

## ABSTRACT

Tree-like branching networks are very common flow or transportation systems from natural evolution. In this study, the optimal structures of tree-like branching networks for minimum flow resistance are analyzed for both laminar and turbulent flow in both smooth and rough pipes. It is found that the dimensionless effective flow resistance under the volume constraint for different flows is sensitive to the geometrical parameters of the structure. The flow resistance of the tree-like branching networks reaches a minimum when the diameter ratio $\beta^*$ satisfies $\beta^* = N^k$, where, $N$ is the bifurcation number $N = 2, 3, 4, \ldots$ and $k$ is a constant. For laminar flow, $k = -1/3$, which is in agreement with the existing Murray's law; for turbulent flow in smooth pipes, $k = -3/7$; for turbulent flow in rough pipes, $k = -7/17$. These results serve as design guidelines of efficient transport and flow systems.


## 1. Introduction

Tree-like branching networks exist widely in nature (such as lungs and vascular systems in mammals, river basins and plants) and have received considerable attention [1–6]. It has been shown that natural tree-like branching networks tend to have a perfect structure [7,8] and often surpass man-made products, in terms of minimal resistance and optimal vascular diameter for driving the blood in mammals and water in plants. This provides inspiration for the design of transport or conversion systems in biological engineering [9,10], chemical engineering [11,12], textile engineering [13,14], microelectronic engineering [15,16] and energy sources recovery [17–19].

The tree-like branching structures of mammalian cardiovascular and respiratory systems are optimum for blood and gas flow, as discussed by Murray et al. in 1926 [20], who found an optimum relationship between the diameter of the parent vessel ($D_k$) and that of two daughter branches ($D_{k+1}$) in the form of $D_{k+1}/D_k = 2^{-1/3}$. The relationship has been verified in recent theoretical analysis and experimental observation [21–23]. And it is now known as Murray's law. Recently, Bejan

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http://dx.doi.org/10.1016/j.physa.2013.08.029

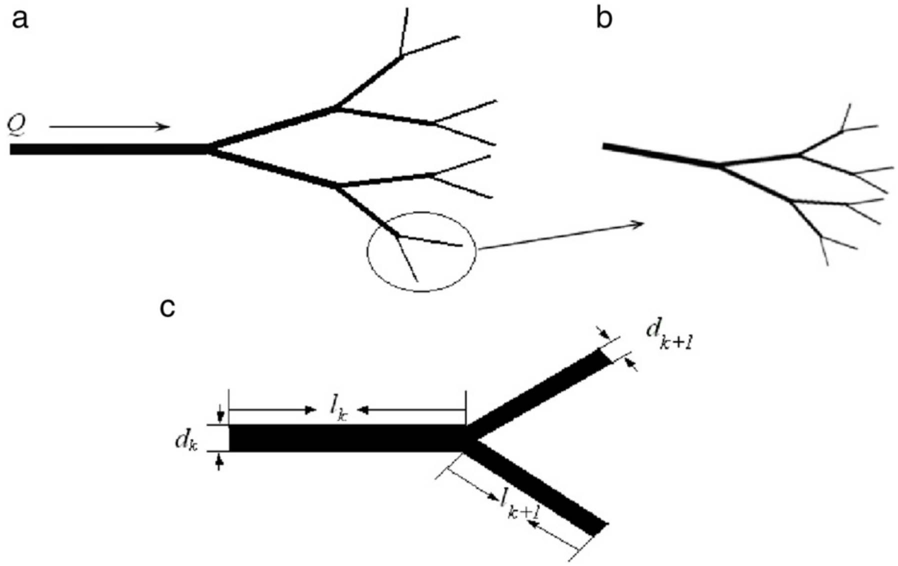


**Fig. 1.** Schematic of self-similar branch structure. The tree-like, spatial fractal (a) has self-similar branches, such that the small-scale structure (b) resembles the large-scale form (a), where the arrow represents the direction of fluid (Q) flow, the self-similar branch structures (a) can be built by repeating a finite number of element structures (c).

[24–31] proposed a "constructal law" to explain the tree-like branching network in nature. The construct law states that, *for a finite-size system to persist in time (to live), it must evolve in such a way that it provides easier access to the imposed currents that flow through it*. Bejan et al. [24–26] showed that a tree-like branching network is an optimal configuration following the constructal law, as it generates minimum entropy for a flow between a source to a volume. Bejan et al. [27] applied the constructal law to the tree-like branching networks with local T-shaped and Y-shaped constructs, and showed that, for fixed total flow volume, there is an optimal diameter ratio of $2^{-1/3}$ and $2^{-3/7}$, respectively. More recently, Xu and Chen et al. [28,29] proposed effective permeability of the whole networks and found tree-like networks could significantly increase the effective permeability of the composites compared to the traditional parallel networks under proper structural parameters. Although optimum tree-like networks have been studied in terms of pumping power requirements by Murray [20] and constructal law [24–37,30–39], the optimum design of the entire network system and the effects of various structural parameters under different types of flows have not been elucidated.

In this paper, we have considered the optimal design of the entire tree-like network for both laminar and turbulent flow. Under the volume constraint of the whole network, we derive a dimensionless expression of effective flow resistance. We discuss the relationship between the dimensionless effective flow resistance and the geometrical parameters of the tree-like branching networks (including diameter ratio, length ratio, branching number). Furthermore, we give an optimal design scheme when the flow resistance of the whole network is minimum.

## 2. Tree-like branching network

A tree-like branching network is a complex structure. In order to minimize the energy dissipated in the system, the network must be a self-similar fractal network that can be space filling [40,41]. In this study, we consider a general Y-shaped tree-like branching network as shown in Fig. 1(a), which can be built by repeating a finite number of elements constructed in the shape as shown in Fig. 1(c). The structure satisfies the self-similar characteristic [40], it is space filling, significantly more variability tolerant than other structures and has an evolutionary advantage [42]. For the structure, every channel is divided into $N$ branches at the next level (e.g., $N = 2$ in Fig. 1) and the branches are of the same geometries. In the present work, we consider that the thickness of the tube wall is sufficiently thin to be negligible; all the ducts are sufficiently slender and the losses at the junctions can be neglected. For a laminar flow through the network, each branch of the network is a smooth cylindrical tube. For a turbulent flow through the network, we consider turbulent flow through rough pipes and smooth pipes.

In order to describe the branching structures, let the length and diameter of a typical branch at some intermediate level $k$ ($k = 0, 1, 2, 3 \ldots$) be $l_k$ and $d_k$, respectively. We further introduce two scale factors, $\beta = d_{k+1}/d_k$ and $\gamma = l_{k+1}/l_k$. Therefore, we can have

$$d_k = d_0\beta^k \quad \text{and} \quad l_k = l_0\gamma^k, \tag{1}$$

where, $l_0$ and $d_0$ are the length and diameter of the 0th branching level, respectively.