Optics and Lasers in Engineering 58 (2014) 60–66

Contents lists available at ScienceDirect

Optics and Lasers in Engineering

journal homepage: www.elsevier.com/locate/optlaseng

Calculation of a surface shape of a pressure actuated membrane liquid lens

Antonín Mikš, Pavel Novák *

Czech Technical University in Prague, Faculty of Civil Engineering, Department of Physics, Thakurova 7, 16629 Prague, Czech Republic

ARTICLE INFO

Article history:
Received 23 November 2013
Received in revised form
11 January 2014
Accepted 28 January 2014
Available online 20 February 2014

Keywords:
Membrane liquid lens
Variable-focus lens
Surface shape calculation
Membrane deflection
Numerical solution

ABSTRACT

Our work is focused on the problem of theoretical evaluation of the surface shape of a variable focus membrane lens. In order to be able to determine the imaging properties of such a lens one has to know the shape of the membrane for a given pressure of the liquid inside the lens chamber with high accuracy. In our work a generalized nonlinear differential equation describing this problem is derived and the solution of this equation using expansion into series and transformation of the problem into constraint optimization problem is proposed. The proposed method enables us to calculate the shape of a membrane lens surface with satisfactory accuracy for further calculations such as e.g. calculation of aberrations of such a lens, imaging properties etc.

1. Introduction

In these days the field of optical elements with variable focal length based on different physical principles is rapidly developing [1–13]. One possible way to realize a variable-focus lens is to use a thin elastic membrane clamped at the edge as one optical surface of a liquid lens of appropriate construction [4–13]. Fig. 1 shows a simple scheme of such a lens.

By changing the volume of the liquid in the chamber of the lens (i.e. changing the pressure p) one can continuously change the shape of the membrane which results in a change of the focal length of the lens. In order to be able to describe imaging properties of such liquid lens it is necessary to know the shape of the membrane as accurately as possible. While in mechanical and construction engineering the error of 5% in the determination of the deflection of a plate (membrane) is satisfactory enough in the field of optics such error is totally unacceptable. Assuming that n and n' are refractive indices of the media in front of and behind the optical refractive surface and δs is the error of the meridian of this surface then the wave aberration δW due to this error is approximately given by the well known relation $\delta W = (n' - n)\delta s$. In case that we choose e.g. $n' = 1.5$, $n = 1$ and we set a requirement on the maximum tolerable change in the wave aberration $\delta W = \lambda/10$, where λ is the wavelength of light, then according to the above mentioned relation the acceptable change in the shape of the optical surface δs is given by $\delta s \le \lambda/5$. Choosing wavelength $\lambda = 0.00055$ mm one obtains $\delta s \le 0.00011$ mm. The shape of the membrane therefore must be determined at minimum with such accuracy.

The problem of elastic deformations of membranes is treated in many books and journal papers e.g. [14–22]. Problem of the plates and membranes with variable thickness is discussed e.g. in [14,15,17,20]. The detailed overview of books dealing with this

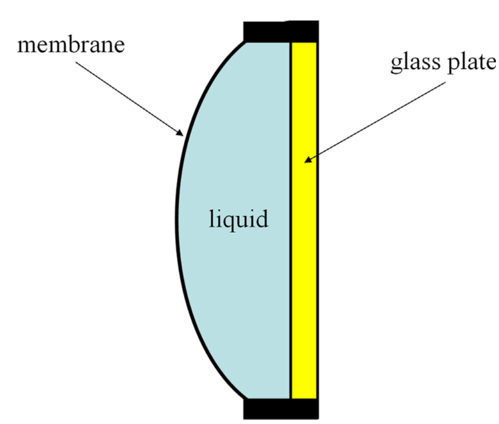

Fig. 1. Scheme of a pressure actuated membrane liquid lens.

* Corresponding author. Tel.: +420 224354435.
E-mail address: pavel.novak.3@fsv.cvut.cz (P. Novák).

http://dx.doi.org/10.1016/j.optlaseng.2014.01.026
0143-8166

problem is given in [18]. Membrane liquid lenses can be also used as elements of microlens matrix arrays (more detailed information can be found e.g. in [23,24]).

In practical applications, it is needed to reduce the influence of gravity on the membrane deformation because it may decrease the lens performance. It was shown that the effect of gravity induced aberration can be negligibly small by a proper selection of the lens diameter, liquid density, membrane elasticity, and fluid pressure [19].

The aim of our work is to calculate the theoretical shape of the surface of the above mentioned membrane liquid lens with the accuracy that would satisfy the requirements on the optical surface quality.

2. Theory of large deflections of thin circular membrane clamped at the edge under uniform pressure

Let us assume the liquid lens shown schematically in Fig. 1 i.e. one of the optical surfaces is a circular membrane of thickness h and diameter $D_{membr}=2a$, which is clamped at the edge. For the calculation of the deflection of the membrane we will start from the relations for calculation of tangential N_t and radial N_r tensile forces per unit length in an absolutely elastic circular membrane under uniform pressure clamped at the edge, it holds [14–17]

$$N_t=\frac{d}{dr}(rN_r)-pr\frac{dw}{dr}, \tag{1}$$

$$N_r\frac{dw}{dr}=-\frac{pr}{2}, \tag{2}$$

where p is the uniform pressure acting on the membrane, w is the lateral deflection of the membrane (deflection in the direction of the z-axis i.e. axis going through the center of the membrane perpendicularly to the membrane surface) and r is the radial distance of the element of the membrane from the z-axis. Assuming linear elastic medium the Hooke's law can be expressed as [14,15].

$$N_t-\mu N_r=Eh\varepsilon_t, \tag{3}$$

$$N_r-\mu N_t=Eh\varepsilon_r, \tag{4}$$

where

$$\varepsilon_t=\frac{u}{r}, \tag{5}$$

$$\varepsilon_r=\frac{du}{dr}+\frac{1}{2}\left(\frac{du}{dr}\right)^2+\frac{1}{2}\left(\frac{dw}{dr}\right)^2, \tag{6}$$

whereas u is the radial displacement, E is the Young's modulus (modulus of elasticity), μ is the Poisson's ratio and h is the membrane thickness. In the classical solution of the problem [17] the second term on the right hand side of Eq. (6) is neglected. The results obtained using the classical theory are therefore affected by this approximation. Let us now focus on the solution of the problem using the exact relations given by Eqs. (3)–(6). By differentiation of Eq. (5) we obtain

$$\frac{du}{dr}=\varepsilon_t+r\frac{d\varepsilon_t}{dr}. \tag{7}$$

By solving Eqs. (1)–(6) after longer calculation we obtain

$$r^2\frac{d^2N_r}{dr^2}+3r\frac{dN_r}{dr}-(2+\mu)pr\frac{dw}{dr}-pr^2\frac{d^2w}{dr^2}+\frac{Eh}{2}\left(\frac{dw}{dr}\right)^2 =-\frac{Eh}{2}\left(\frac{du}{dr}\right)^2. \tag{8}$$

By using the following denotation

$$N=\frac{N_r}{pa},\ q=\frac{pa}{Eh},\ \rho=r/a, \tag{9}$$

and employing Eq. (2) we can rewrite Eq. (8) in a form

$$N^2\left(\rho^2\frac{d^2N}{d\rho^2}+3\rho\frac{dN}{d\rho}\right)-\frac{\rho^3}{2}\frac{dN}{d\rho}+(3+\mu)\frac{\rho^2}{2}N+\frac{\rho^2}{8q}=-\frac{N^2}{2q}\left(\frac{du}{dr}\right)^2, \tag{10}$$

where the right hand side can be further expressed as

$$\frac{N^2}{2q}\left(\frac{du}{dr}\right)^2=\frac{q}{2N^2}\left\{N^2\left[\rho^2\frac{d^2N}{d\rho^2}+(3-\mu)\rho\frac{dN}{d\rho}+(1-\mu)N\right]-\frac{\rho^3}{2}\frac{dN}{d\rho}+\frac{3}{2}\rho^2N\right\}^2. \tag{11}$$

Eq. (10) is a generalization of classical solution given in [17]. By substitution of Eq. (11) into Eq. (10) we obtain the generalized differential equation for function N. The following boundary conditions apply for a circular membrane of the diameter $D_{membr}=2a$ clamped at the edge

$$w(r)|_{r=a}=0,\ u(r)|_{r=a}=0, \tag{12}$$

By setting the right hand side of Eq. (10) to zero one obtains the classical solution given e.g. in [17]. By solving of the generalized Eq. (10) we obtain the function $N(\rho)$ and the deflection of the membrane $w(\rho)$ is then obtained by solving Eq. (2).

3. Numerical solution of the shape of the membrane lens surface

Different approaches can be used in order to find the solution of the nonlinear differential Eq. (10). There exist a lot of numerical methods for solving differential equations [14–20]. A very good overview of these methods applied to the problem of thin plates and membranes is given in [14,15,20] where also the problem of the appropriate boundary conditions (clamping), edge effects etc. is discussed. One of these is a *series method* which is relatively simple and it gives good results. General principle of this method is the following. The sought solution $f(\xi)$ of given differential equation is expressed using expansion in a series

$$f(\xi)=\sum_{i=0}^{K}c_i\,g_i(\xi), \tag{13}$$

where functions $g_i(\xi)$ are appropriately chosen to suit given problem [14–20]. For the appropriate choice of these functions it is good to know some other information resulting from the nature of the solved problem e.g. that the function $f(\xi)$ is symmetrical etc. As it is known every function can be approximated using polynomial expansion (Taylor series) i.e. one can use simply the power series expansion as function $g_i(\xi)$. Sometimes it is advantageous to use polynomials orthogonal on given area i.e. for 2D rectangular area one can use e.g. 2D Legendre polynomials, for circular area one can use Zernike polynomials. In case of some symmetry of the problem one can further restrict the expansion only to the symmetrical terms of expansion. In our case the power series expansion with even powers only is assumed due to the rotational symmetry (circular shape, constant pressure). The unknown quantities that are to be determined are then the coefficients c_i. The determination of these coefficients can then be performed numerically using optimization techniques. For example in our case we know that the shape of the membrane is symmetrical with respect to z-axis therefore we can express