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# Characteristic polynomial theory of two-stage phase shifting algorithms

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ABSTRACT

Two-stage phase shifting algorithms make possible to directly recover the sum or the difference of the optical phase of two different fringe patterns. These algorithms can be built by combining the known phase shifting algorithms in a non-linear way. In this work, we associate a two-dimensional characteristic polynomial to each two-stage phase shifting algorithm. This enables us to qualitatively compare their behaviour against the main systematic error sources, by means of an analysis protocol like that used for phase shifting algorithms. We show that this tool allows to understand the propagation of properties from precursor phase shifting algorithms to new evaluation algorithms built from them. As an experimental application, a wavefront distortion evaluation in differential phase-shifting interferometry is presented.



## 1. Introduction

A great variety of metrological applications [1] require the use of phase shifting interferometry [2]. In it, $M$ irradiance values equally shifted in phase [3], $\alpha_m=2\pi(m-1)/M$, and with $k$-harmonic components $a_k$

$$s_m(\mathbf{r},\phi,\alpha_m)=\sum_{k=0}^{\infty}a_k(\mathbf{r})\cos[k(\phi+\alpha_m)] \tag{1}$$

are linearly combined in the argument of an arctan function [4] to recover the phase of the pattern

$$\phi(\mathbf{r})=\arctan\frac{\sum_{m=1}^{M}n_m s_m(\mathbf{r},\phi,\alpha_m)}{\sum_{m=1}^{M}d_m s_m(\mathbf{r},\phi,\alpha_m)}=\arctan\frac{\sum_{m=1}^{M}h_N(u_{1m})s_m(\mathbf{r},\phi,\alpha_m)}{\sum_{m=1}^{M}h_D(u_{1m})s_m(\mathbf{r},\phi,\alpha_m)} \tag{2}$$

where $(n_m,d_m)$ are the sampling amplitudes that define each phase shifting algorithm (PSA). This process can also be expressed as a discretization of the irradiance signal $s(\mathbf{r},\phi)$ with sampling functions $h_{N,D}(u_1)$ [5,6] – where $u_1$ is the spatial or temporal variable – at each sampling point $u_1=u_{1m}$, where $\alpha_m=2\pi f_s u_{1m}$ with $f_s$ its reference frequency. The Fourier transform (FT) of these sampling functions are called characteristic spectra (CS), $H_{N,D}(f_1)=\mathrm{FT}[h_{N,D}(u_1)]$, and their amplitudes provide a qualitative vision of the main errors sources of the PSAs. But this frequency analysis, carried out independently with the two sampling functions, can lead to proportional spectral representations of each rotation of the same PSA with identical compensatory properties and with a difference, in general, connected to a constant phase term that is irrelevant in many situations. This non-evident similarity between shifted PSAs can be sidestepped [7–12] using a more complete analysis considering the combined CS of the PSA defined as

$$H=H_D(f_1)+jH_N(f_1) \tag{3}$$

An alternative, and equivalent, analysis of sensitivities is obtained relating the calculation of the optical phase of a PSA to an associated complex number $V(\phi)$. So Eq. (2) can be rewritten as

$$V(\phi)=\sum_{m=1}^{M}(d_m+jn_m)s_m(\mathbf{r},\phi,u_m)=\sum_{k=-\infty}^{\infty}\left[a_k e^{jk\phi}\sum_{m=1}^{M}(d_m+jn_m)e^{jk\alpha(m-1)}\right]=\sum_{k=-\infty}^{\infty}[a_k e^{jk\phi}P(e^{jk\alpha})] \tag{4}$$

with additional phase $\alpha=\alpha_{m+1}-\alpha_m=\alpha_m/(m-1)$ being $P[\exp(jk\alpha)]$ the characteristic polynomial (CP) of the PSA. The multiplicity and localisation of the roots of the CP inform about the main systematic error sources in PSAs, such as the presence of undesired harmonics in the recovered signal or phase shift errors. So, if we consider the $k$-coefficients $a_k$ with $k>1$ non zero and the additional phase disturbed in the form

$${}^{E}\alpha_m=\alpha_m+E\alpha_m=\alpha_m+\sum_{q=1}^{\infty}\varepsilon_q\frac{\alpha_m^q}{q\pi^{q-1}} \tag{5}$$

with the error $E\alpha_m$ quantified by the $q$-th term $\varepsilon_q$, we can express the Eq. (1) affected by both errors:

$$s_m(\mathbf{r},\phi,{}^{E}\alpha_m)=\sum_{k=0}^{\infty}a_k(\mathbf{r})\cos[k(\phi+{}^{E}\alpha_m)]=\sum_{k=-\infty}^{\infty}\frac{a_k(\mathbf{r})}{2}e^{jk(\phi+{}^{E}\alpha_m)} \tag{6}$$

Thus [13–18], for instance, elimination of local average irradiance effects for the fringe pattern requires the cancelling of the CP at $k=0$, while the insensitivity to an undesired $k$-harmonic $(k>1)$ demands that $P[\exp(jk\alpha)]$ was cancelled at this frequency, which, in

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factorisation terms, means that $\exp(jk\alpha)$ and $\exp(-jk\alpha)$ must be roots of the CP. Regarding phase detuning errors (first order errors in the phase step), the double root of $\exp(-j\alpha)$ must appear, and not that for $\exp(j\alpha)$ since the fundamental harmonic must be detected.

In any case, the analysis with the combined CS and the CP are directly related [18] and both of them obtain coinciding results so

$$P(e^{jk\alpha}) = \sum_{m=1}^{M}(d_m + jn_m)e^{(m-1)jk\alpha} = \sum_{m=1}^{M} d_m e^{j\alpha_m \frac{f_1}{f_s}} + j\sum_{m=1}^{M} n_m e^{j\alpha_m \frac{f_1}{f_s}} = H_D(-f_1) + jH_N(-f_1) \quad (7)$$

But it can also be the case that the information of the measurand [19–27] was function of the sum or the difference of the optical phase between an original fringe pattern $s(\mathbf{r},\phi)$ and another modified one $t(\mathbf{r},\phi+\Delta\phi)$ with irradiance values weighted with $g$-coefficients $b_g$:

$$t_p(\mathbf{r},\phi+\Delta\phi,\beta_p) = \sum_{g=0}^{\infty} b_g(\mathbf{r})\cos[g(\phi+\Delta\phi+\beta_p)] \quad (8)$$

Likewise, the phase $\phi(\mathbf{r})+\Delta\phi(\mathbf{r})$ is also obtained by the linear combination of $P$ irradiance values

$$\phi(\mathbf{r})+\Delta\phi(\mathbf{r}) = \arctan\frac{\sum_{p=1}^{P} n_p t_p(\mathbf{r},\phi+\Delta\phi,\beta_p)}{\sum_{p=1}^{P} d_p t_p(\mathbf{r},\phi+\Delta\phi,\beta_p)} = \arctan\frac{\sum_{p=1}^{P} h_N(u_{2p}) t_p(\mathbf{r},\phi+\Delta\phi,\beta_p)}{\sum_{p=1}^{P} h_D(u_{2p}) t_p(\mathbf{r},\phi+\Delta\phi,\beta_p)} \quad (9)$$

with $(n_p,d_p)$ the sampling amplitudes at the point analysed $u_2=u_{2p}$ as $\beta_p=2\pi f_t u_{2p}$ where $f_t$ is its reference frequency and the sampling functions are related with the CS of the modified pattern as $H_{N,D}(f_2)=\mathrm{FT}[h_{N,D}(u_2)]$. We can associate the phase of the pattern with the phase of the complex function $V(\phi+\Delta\phi)$. So, the sensitivity of the modified pattern can also be given by the behaviour of its CP $P[\exp(jg\beta)]$

$$\mathrm{V}(\phi+\Delta\phi) = \sum_{p=1}^{P}(d_p+jn_p)t_p(\mathbf{r},\phi+\Delta\phi,u_p) = \sum_{g=-\infty}^{\infty}\left[b_g e^{jg(\phi+\Delta\phi)}\sum_{p=1}^{P}(d_p+jn_p)e^{jg\beta(p-1)}\right] = \sum_{g=-\infty}^{\infty}[b_g e^{jg(\phi+\Delta\phi)}P(e^{jg\beta})] \quad (10)$$

where $\beta=\beta_{p+1}-\beta_p=\beta_p/(p-1)$. We can also assume the presence of $g$ undesired harmonics $b_g$ and errors in the phase shifter in the form

$$^{E}\beta_p = \beta_p + E\beta_p = \beta_p + \sum_{r=1}^{\infty}\chi_r\frac{\beta_p^r}{r\pi^{r-1}} \quad (11)$$

quantifying $X_r$ the magnitude of $r$-th order. So the modified pattern, Eq. (8), is disturbed in the form

$$t_p(\mathbf{r},\phi+\Delta\phi,{}^{E}\beta_p) = \sum_{g=0}^{\infty} b_g(\mathbf{r})\cos[g(\phi+\Delta\phi+{}^{E}\beta_p)] = \sum_{g=-\infty}^{\infty}\frac{b_g(\mathbf{r})}{2}e^{jg(\phi+\Delta\phi+{}^{E}\beta_p)} \quad (12)$$

Thus, the last step to recover the additive phase or the difference phase between the original and the modified patterns would be to add or subtract, respectively, the unwrapped phases [28] obtained with Eqs. (2) and (9). If a characterisation procedure was required the calculation of the combined CS and/or the CP of each pattern would also be necessary. All this laborious process can be simplified if two-stage phase shifting algorithms (TSPSAs) [29–36] are employed:

$$[\phi(\mathbf{r})+\Delta\phi(\mathbf{r})] \pm \phi(\mathbf{r}) = \arctan\frac{\sum_{m=1}^{M} d_m s_m(\mathbf{r},\phi,\alpha_m)\sum_{p=1}^{P} n_p t_p(\mathbf{r},\phi+\Delta\phi,\beta_p) \pm \sum_{m=1}^{M} n_m s_m(\mathbf{r},\phi,\alpha_m)\sum_{p=1}^{P} d_p t_p(\mathbf{r},\phi+\Delta\phi,\beta_p)}{\mp\sum_{m=1}^{M} n_m s_m(\mathbf{r},\phi,\alpha_m)\sum_{p=1}^{P} n_p t_p(\mathbf{r},\phi+\Delta\phi,\beta_p) + \sum_{m=1}^{M} d_m s_m(\mathbf{r},\phi,\alpha_m)\sum_{p=1}^{P} d_p t_p(\mathbf{r},\phi+\Delta\phi,\beta_p)} \quad (13)$$

This two-dimensional analysis combine the numerator and the denominator of the precursor PSAs, Eqs. (2) and (9), to recover the phase sum $2\phi(\mathbf{r})+\Delta\phi(\mathbf{r})$ (upper sign) or the phase difference $\Delta\phi(\mathbf{r})$ (lower sign) in a unique calculation. Moreover, they can provide directly continuous values of the phase difference $\Delta\phi(\mathbf{r})$ when a period is not completed in the whole pattern area and thus avoid the unwrapping process. The advantages of the use of TSPSAs depend on the particular application. For example, in phase-shifted ESPI measurements [21] the subtraction in the fringe pattern domain, before the arctangent is applied, has the noticeable advantage of facilitating the application of smoothing routines to improve the contrast of the correlation fringes; or in multi-wavelength interferometry, the fringe pattern domain calculation reduces the sensitivity to random irradiance fluctuations and facilities its ability to measure diffuse, as well as reflective, surfaces [20]. In previous works [32–36], we have shown the good behaviour of TSPSAs by means of quantitative and qualitative analysis of their main associated errors. All these techniques supply the same information with the obvious difference between a quantitative and a qualitative research. Thus, the Taylor development [32] or the Monte Carlo method [33,36] obtain a curve of the error whereas the Fourier analysis [34] can only inform about the presence or absence of the error. The procedures are independent, the choice of either one depends on the capabilities of the researcher. Here, we aim to complete the qualitative characterisation of these algorithms using as a tool a two-dimensional characteristic polynomial (TDCP) that provides information about the main sensitivities of the TSPSAs and relates them with the sensitivities of their precursor PSAs.

## 2. Two-dimensional characteristic polynomial

The generic equation of a TSPSA, Eq. (13), can also be expressed as a quantification process [34]

$$[\phi(\mathbf{r})+\Delta\phi(\mathbf{r})] \pm \phi(\mathbf{r}) = \arctan\frac{\sum_{m=1}^{M}\sum_{p=1}^{P} n_{m,p}s_m(\mathbf{r},\phi,\alpha_m)t_p(\mathbf{r},\phi+\Delta\phi,\beta_p)}{\sum_{m=1}^{M}\sum_{p=1}^{P} d_{m,p}s_m(\mathbf{r},\phi,\alpha_m)t_p(\mathbf{r},\phi+\Delta\phi,\beta_p)} = \arctan\frac{\sum_{m=1}^{M}\sum_{p=1}^{P} h_N(u_{1m},u_{2p})s_m(\mathbf{r},\phi,\alpha_m)t_p(\mathbf{r},\phi+\Delta\phi,\beta_p)}{\sum_{m=1}^{M}\sum_{p=1}^{P} h_D(u_{1m},u_{2p})s_m(\mathbf{r},\phi,\alpha_m)t_p(\mathbf{r},\phi+\Delta\phi,\beta_p)} \quad (14)$$

where $(n_{m,p},d_{m,p})$ are the sampling amplitudes of the auxiliary mathematical function $s(\mathbf{r},\phi)t(\mathbf{r},\phi+\Delta\phi)$, which are precisely the values that make good recovery of either the phase sum, when $n_{m,p}=\sin(\alpha_m+\beta_p)$ and $d_{m,p}=\cos(\alpha_m+\beta_p)$, or the phase difference, $n_{m,p}=\sin(\alpha_m-\beta_p)$ and $d_{m,p}=\cos(\alpha_m-\beta_p)$. Thus, the goodness of a TSPSA is also determined by its sensitivities to the main error sources that are given in the reciprocal space by their two-dimensional CS (TDCS) $H_{N,D}(f_1,f_2)=\mathrm{FT}[h_{N,D}(u_1,u_2)]$. A similar qualitative characterisation can be carried out by associating a complex number $V[(\phi+\Delta\phi)\pm\phi]$ to the calculation of the phase,