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Matérn cross-covariance functions for bivariate spatio-temporal random fields

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A B S T R A C T

Spatio-temporal processes involving more than one variable emerge in various fields. Any serious attempt of statistical inference and prediction for multivariate data require knowledge about the dependency structures within and across variables. In this work, we provide general conditions leading to positive semi-definiteness of the overall matrix-valued covariance functions. Both the marginal and cross-covariance functions belong to a generally non-separable Matérn class spatio-temporal covariance functions, but with possibly different scale, smoothness and space–time separability parameters. The main focus of this work is on bivariate spatio-temporal random fields. As an illustration, the model is fitted on a set of bivariate air pollution data.

1. Introduction

Applications of multivariate spatial and spatio-temporal statistics can be found in various disciplines. Modelling of the dependency structure within and between variables is curial to statistical inferences and predictions. In the literature, modelling of cross-covariance functions under the purely spatial framework has attracted vast amount of interest. Genton and Kleiber (2015), Journel and Huijbregts (1978) and Wackernagel (2003) provided excellent reviews on cross-covariance functions for multivariate spatial data.

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In building cross-covariance functions for multivariate spatial data, one of the traditional methods is the linear coregionalization model (LCM) (Goulard and Voltz, 1992). Under LCM, the multivariate cross-covariance function is a linear combination of independent univariate valid covariance functions. Further developments of the LCM method can be found in Gelfand et al. (2004), Schmidt and Gelfand (2003) and Zhang (2007). For LCM in the space–time setting, one may consult De Iaco et al. (2003). Another popular approach is the convolution method which usually depends on numerical integrations, interested readers may refer to Du and Ma (2013), Fuentes and Reich (2013), Majumdar and Gelfand (2007), Majumdar et al. (2010), and Ver Hoef and Barry (1998). More recently, Apanasovich and Genton (2010) proposed cross-covariance functions using latent dimensions which are possible to be extended to the spatio-temporal setting. However, the interpretation of such a model is not straightforward. Furthermore, Kleiber and Genton (2013) introduced cross-covariance functions with spatially varying cross-correlation coefficients.

Out of the many available classes of covariance functions, the Matérn class (Guttorp and Gneiting, 2006; Matérn, 1986) is a popular choice among scholars in recent years, due to the strong support from Stein (1999) and its flexibility as described later. The Matérn class covariance function is defined as $\sigma^2 M_\nu(h;\alpha)$ where σ^2 is the variance and

$$M_\nu(h;\alpha) = \frac{(\alpha h)^\nu \mathcal{K}_\nu(\alpha h)}{2^{\nu-1}\Gamma(\nu)} \tag{1}$$

is the spatial correlation function evaluated at spatial distance $h = \|\boldsymbol{s}_1 - \boldsymbol{s}_2\|$. In (1), $\alpha > 0$ is the scale parameter governing the decay of spatial correlation, ν is the smoothness parameter and $\mathcal{K}_\nu(\cdot)$ is the modified Bessel function of the second kind, see Abramowitz and Stegun (1972, Ch. 9) for further details. When $\nu = 0.5$, the Matérn covariance function reduces to the exponential covariance function. Parallel to the strong interest in the univariate case, multivariate cross-covariance models that are in the Matérn class have been developed in the purely spatial setting. In Gneiting et al. (2010), the bivariate Matérn covariance models were discussed. In particular, conditions leading to valid bivariate cross-covariance models were given. The special case when exponential marginal and cross covariance functions are employed was studied by Chilès and Delfiner (1999). For more than two variables, Apanasovich et al. (2012) introduced ways to construct valid Matérn cross-covariance models for any number of variables. Kleiber and Nychka (2012) further extended the idea to model multivariate spatial processes which are nonstationary. Some other approach in building bivariate cross-covariance models can be found in Fanshawe and Diggle (2012). However, these works have been focusing on the multivariate spatial field, leaving the problems regarding the multivariate spatio-temporal fields open.

We consider the bivariate spatio-temporal random field $\boldsymbol{X}(\boldsymbol{s},t) = (X_1(\boldsymbol{s},t), X_2(\boldsymbol{s},t))^\top$ defined on a spatio-temporal region $\mathcal{D} = \mathcal{D}_S \times \mathcal{D}_T \subset \mathbb{R}^d \times \mathbb{R}$ where each $X_i(\boldsymbol{s},t)$ is the ith variable at location $\boldsymbol{s} \in \mathbb{R}^d$ and time $t \in \mathbb{R}$, for $i = 1, 2$. Here, it is assumed that the mean of $\boldsymbol{X}(\boldsymbol{s},t)$ is zero and the cross-covariance function is isotropic so that

$$\mathrm{Cov}(X_i(\boldsymbol{s}_1,t_1), X_j(\boldsymbol{s}_2,t_2)) = C_{ij}(h,u), \quad i,j = 1,2, \tag{2}$$

where $h = \|\boldsymbol{s}_1 - \boldsymbol{s}_2\|$, where $\|\cdot\|$ denotes the Euclidean norm and $u = |t_1 - t_2|$. Furthermore, full symmetry (Gneiting et al., 2007) is assumed so that $C_{ij}(h,u) = C_{ji}(h,u)$ for $i \neq j$. Full symmetry and isotropy are often restricted assumptions. Yet, they are always the building blocks for more general models. A spatio-temporal covariance function $C_{ST}(h,u)$ is called separable if it can be written as $C_S(h)C_T(u)$ where C_S is a purely spatial covariance function and $C_T(u)$ is a purely temporal covariance function (Cressie and Huang, 1999; Gneiting, 2002). If $\boldsymbol{\Sigma}$ is the covariance matrix of the random vector

$$\left(\boldsymbol{X}(\boldsymbol{s}_1,t_1)^\top, \ldots, \boldsymbol{X}(\boldsymbol{s}_1,t_T)^\top, \ldots, \boldsymbol{X}(\boldsymbol{s}_K,t_1), \ldots, \boldsymbol{X}(\boldsymbol{s}_K,t_T)^\top\right)^\top,$$

it must be positive semi-definite (p.s.d.) so that the condition $\boldsymbol{c}^\top \boldsymbol{\Sigma} \boldsymbol{c} \geq 0$ is satisfied for any real-valued vector $\boldsymbol{c}$ of dimension $2KT$. With separable spatio-temporal covariance functions, valid cross-covariance functions can be constructed easily by making use of the Schur's product theorem (Horn and Johnson, 1990). Yet, separable models are often insufficient in practice, as noted by De Iaco and Posa (2013). Hence, our goal is to introduce valid generally non-separable cross-covariance functions such that all the marginal and cross-covariance functions belong to the Matérn class.