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Rational spectral density models for lattice data

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Dedicated to the memory of J.E. Besag, 1945–2010

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ABSTRACT

Conditional autoregressive CAR models, possibly with added noise, unilateral ARMA models, and directly specified correlation DC models, are widely used classes of spatial models. In this paper, we consider their generalisation to all models with a rational spectral density function. These models allow a wider range of correlation behaviour, and can provide adequate fits to data with fewer parameters. Some theoretical properties are presented, and comparisons made with CAR correlations. Some methods for estimation are discussed, and fits to some real data are compared.

1. Introduction

Large amounts of essentially-continuous spatial data are associated with the nodes or interiors of a regular rectangular lattice. Examples include pixellated images which occur in many different applications, regularly-sampled spatial data, and many agricultural field trials. Different types of models have been proposed for analysing such data. Four main classes are: (i) those with a directly specified correlation structure, such as those used in geostatistics; (ii) those specified by a generating model involving 'past' (using some site ordering) values and uncorrelated innovations—unilateral (or causal) autoregressive-moving average ARMA models; (iii) those specified by a formal equation involving 'past' and 'future' values and uncorrelated errors – simultaneous autoregressions SAR; and (iv) those that specify the conditional distribution at each site given the values at all other sites –

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conditional autoregressive CAR models. An extension to SAR and CAR models adds an independent noise term.

We (essentially) only consider univariate stationary Gaussian models here—i.e. they have finite variance, and the correlation between the observations at two sites only depends on the relative positions of the two sites. The assumption of Normality for analysing a data set is convenient and often reasonable (perhaps after transformation). We assume models are defined on an infinite regular rectangular lattice, and applied to data on a complete finite lattice. Then a CAR, also known as a Gauss–Markov random field, is defined by its conditional means. Every unilateral AR, and every SAR, is equivalent to a CAR (in the sense of having the same correlation structure).

Since many unilateral ARMA models depend on the choice of site ordering, they can appear arbitrary, but they can have some useful properties, especially if the model is separable (i.e. the correlation function is a product of lower-dimensional correlation functions). For example, simulation and likelihood evaluation can be simple. SAR models have some severe logical difficulties (e.g. the errors are correlated with all the observations, and in general the parameters are not uniquely determined).

In general, the correlation structure of a CAR model is hard to determine (except numerically), but the inverse correlations are directly specified. On a planar lattice, this gives most, but not all, elements of the inverse dispersion matrix which is required for Gaussian maximum likelihood estimation—see Section 6.1. Another computational difficulty with many CAR models used in practice is that unless the dependence is weak, the parameters are usually very close to the stationarity boundary. Since the correlations of a CAR can decay very slowly from 1, the extension to the noisy CAR has been found useful.

Geostatistical models are defined for continuous space, and are widely used for analysing data defined on irregular sites or regions—see, for example, Cressie (1993, Section 2.3.1). On a regular lattice, they and other models specifying the correlations usually have the drawback that the inverse dispersion matrix does not have a simple form. These lattice models, and moving-average models, in general have an infinite CAR representation.

In time series, the extension of AR and MA models to the ARMA models, which have a rational spectral density, has been extremely useful. The unilateral ARMA models on a lattice have a rational spectral density, but are only a subset of all the possibilities. In this paper, we consider the generalisation of finite CARs, unilateral ARMAs, and finite DCs (directly-specified correlation models for which the correlations are 0 outside a neighbourhood of the origin), to all models with a rational spectral density function—RSDs. The RSDs in general have more possible correlation structures, and can give more parsimonious fits to data. Also, they are less likely than CARs to have the estimated parameters very close to the stationarity boundary.

After reviewing the standard lattice models in Section 2, the RSD model is defined and some of its properties discussed in Section 3. We show how RSDs can arise from operations on CARs in Section 4, and compare their correlations with those of CARs in Section 5. Section 6 discusses how standard methods for model fitting and identification can be extended to RSDs, and in Section 7, fits of models to real data are compared. Note that the results in Sections 2–5 only depend on second-order properties, and so hold for any distribution. For convenience, we mainly refer here to Guyon (1995), Cressie (1993) and Rue and Held (2005) for known results.

2. Lattice models

In this section we discuss the usual lattice models. We begin with some definitions in Section 2.1, and then review the usual AR, SAR, MA, ARMA, CAR, and DC lattice models in Sections 2.2 and 2.3. Some extensions are given in Section 2.4.

2.1. Preliminaries

Suppose that $\boldsymbol{t}$, $\boldsymbol{u}$, $\boldsymbol{z}$ and $\boldsymbol{\lambda}$ are d-dimensional vectors, and assume that $\{x(\boldsymbol{t}), \boldsymbol{t} \in \mathbb{Z}^d\}$ is a second-order stationary random field on the regular rectangular lattice, with mean zero, autocovariance function $R_x(\boldsymbol{u}) = \mathrm{Cov}\{x(\boldsymbol{t}), x(\boldsymbol{t}+\boldsymbol{u})\}$, and autocorrelation function $r_x(\boldsymbol{u}) = R_x(\boldsymbol{u})/\sigma_x^2$, where $\sigma_x^2 =$