Spatial Statistics 6 (2013) 109–117

Contents lists available at ScienceDirect

Spatial Statistics

journal homepage: www.elsevier.com/locate/spasta

Faà di Bruno's formula and spatial cluster modelling☆

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ARTICLE INFO

Article history:
Received 14 May 2013
Accepted 30 September 2013
Available online 16 October 2013

Keywords:
Faà di Bruno's formula
Spatial cluster modelling
Point processes
Variational calculus
Chain differentials

ABSTRACT

The probability generating functional (p.g.fl.) provides a useful means of compactly representing point process models. Cluster processes can be described through the composition of p.g.fl.s, and factorial moment measures and Janossy measures can be recovered from the p.g.fl. using variational derivatives. This article describes the application of a recent result in variational calculus, a generalisation of Faà di Bruno's formula, to determine such results for cluster processes.

1. Introduction

Neyman and Scott proposed the stochastic process of clustering as a mathematical model of galaxies (Neyman and Scott, 1958). Since then spatial clustering models have been investigated in various applications. A collection of articles on spatial cluster modelling reviewed current approaches for statistical inference of spatial and spatial cluster processes and their applications (Lawson and Denison, 2002).

The probability generating functional (Moyal, 1962; Bogolyubov, 1946), p.g.fl., provides a means of uniquely characterising a point process. In a similar way to the probability generating function, the probability measures and factorial moment measures of point processes can be found from the p.g.fl. by differentiation, with Gâteaux differentials (Gâteaux, 1919). The probability generating

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[1] Daniel Clark is a Royal Academy of Engineering/EPSRC Research Fellow.
[2] Jeremie Houssineau has a Ph.D. scholarship sponsored by DCNS and a tuition fees scholarship by Heriot–Watt University.

2211-6753/$ – see front matter
http://dx.doi.org/10.1016/j.spasta.2013.09.002

functional is well known within the point process literature (Daley and Vere-Jones, 2003, p. 15; Cressie, 1991, p. 627; Moller and Waagepetersen, 2003, p. 9; Cox and Isham, 1980, p. 38), yet the approach of functional differentiation is rarely used since it results in a combinatorial number of terms. To highlight this point, we note that cluster processes, such as the model by Neyman and Scott (1958), can be specified simply through the composition of p.g.fl.s, e.g. Cox and Isham (1980, p. 76), Daley and Vere-Jones (2003, p. 178) and Moyal (1962), yet the Janossy measure for arbitrary point processes does not appear to have been determined using this method. Specific parameterisations with Neyman–Scott processes have been studied, though researchers often prefer to work directly with the measures, e.g. Ripley (1988, p. 5), van Lieshout and Baddeley (2002), van Lieshout (2000, p. 140) and Illian et al. (2008, p. 368), rather than with the p.g.fl. representation.

In the aerospace and signal processing literature, the approach has become popular for deriving algorithms for tracking multiple targets from radar, following Mahler's method for Bayesian estimation with point processes (Mahler, 2003, 2007). Practical applications of these techniques were made possible through the development of sequential Monte Carlo (Vo et al., 2005) and Gaussian mixture implementations (Vo and Ma, 2006; Vo et al., 2007). To determine Bayes' theorem for point processes, Mahler (2003) proposed the use of functional derivatives of the probability generating functional. This approach often involves finding the parameterised form of the updated process, and proving its correctness by induction. The process can be quite involved and needs to be applied for each model (Mahler, 2003, 2007, 2009a,b; Swain and Clark, 2010). The construction of the models often involves composition of basic models, whose derivatives are easy to find, yet when composed, their form for higher derivatives becomes more unclear. In this paper we circumvent this problem by using a recently derived tool from variational calculus, Faà di Bruno's formula for variational calculus (Clark and Houssineau, 2013). We use this approach to determine the Janossy measures and factorial moment measures of cluster processes and illustrate the approach through an example with Matérn cluster processes.

2. Variational calculus and the higher-order chain rule

This section describes differentials and the general form of Faà di Bruno's formula required to determine the results in the following section. We adopt a restricted form of Gâteaux differential, known as the chain differential (Bernhard, 2005), in order that a general chain rule can be determined (Clark and Houssineau, 2013). Following this, we describe the general higher-order chain rule.

Definition 2.1 (*Chain Differential, from Bernhard, 2005*)**.** The function $f : X \to Y$, where X and Y are normed spaces, has a *chain differential* $\delta f(x; \eta)$ at $x \in X$ in the direction $\eta \in X$ if, for any sequence $\eta_n \to \eta$ in X, and any sequence of real numbers $\theta_n \to 0$, it holds that

$$\delta f(x; \eta) = \lim_{n\to\infty} \frac{1}{\theta_n} \left(f(x + \theta_n \eta_n) - f(x)\right). \tag{1}$$

The nth-order chain differential can be defined recursively as

$$\delta^n f(x; \eta_1, \ldots, \eta_n) = \delta\left(\delta^{n-1} f(x; \eta_1, \ldots, \eta_{n-1}); \eta_n\right). \tag{2}$$

Applying nth-order chain differentials on composite functions can be an extremely laborious process since it involves determining the result for each choice of function and proving the result by induction. For ordinary derivatives, the general higher-order chain rule is normally attributed to Faà di Bruno (1855). The following result generalises Faà di Bruno's formula to chain differentials.

Theorem 2.1 (*General Higher-Order Chain Rule, from Clark and Houssineau, 2013*)**.** *Let T, U and V be normed spaces. Assume that $g : T \to U$ has higher order chain differentials in any number of directions in the set $\{\eta_1, \ldots, \eta_n\}$, with $\eta_1, \ldots, \eta_n \in T$ and that $f : U \to V$ has higher order chain differentials in any number of directions in the set $\{\delta^m g(x; S_m)\}_{m=1:n}$, $S_m \subseteq \{\eta_1, \ldots, \eta_n\}$. Assuming additionally that for all $1 \le m \le n$, $\delta^m f(y; \xi_1, \ldots, \xi_m)$ is continuous on an open set $\Omega \subseteq Y^{m+1}$ and linear with respect to the*