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Sufficient conditions for the stability of a class of second order systems[☆]

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ABSTRACT

This paper provides sufficient conditions to assess the stability and Input-to-State stability of a class of second-order systems by only looking at the structure of the dynamic equations. These results are proven by using the Variable Gradient Method to build suitable Lyapunov functions. The paper includes a number of relevant examples that highlight the value of the contribution.

1. Introduction

A large number of physical systems exhibit a second-order dynamical behavior. Among others, mechanical systems, electromagnetic circuits and hydraulic networks fall in such category. For this class of systems, it is therefore of interest to derive sufficient conditions ensuring stability and asymptotic stability [1]. In [2], it is shown that the origin of the system $\ddot{x} = -h(x) - a\dot{x}$ is asymptotically stable if $h(x)$ is an odd function and $a > 0$. Likewise, in [3], it is proven that the origin of the system $M\ddot{x} + D\dot{x} + f(x) = 0$ is asymptotically stable if M and D are positive definite matrices and $f(x)$ is the gradient of a conservative field. In [4], the authors provide some sufficient conditions for the stability of a system obtained via the Euler–Lagrange theorem. Results in [5] prove the stability of a system in the form $\ddot{x} + g(t)\dot{x} + f(x) = 0$, where $f(x)$ is an odd function and $g(t)$ is a positive and time-varying damping term. Although these results are quite general, there are still many practically relevant second order systems that fall outside the above classes and require an *ad-hoc* stability analysis. Examples include robots controlled with a saturated *PD* with gravity compensation [6], and the proximate time-optimal control of a double integrator [7]. Moreover, most of these results were proven using *weak* Lyapunov functions coupled with LaSalle's invariance principle, thus preventing Input-to-State Stability (ISS) analysis.

The objective of this paper is twofold. First, it extends the class of scalar second-order systems for which sufficient conditions of asymptotic stability can be provided. Second, it provides stronger stability results, in particular exponential and input-to-state stability, by deriving *strict* Lyapunov functions. These objectives are achieved by building suitable Lyapunov functions by means of the Variable Gradient Method (VGM) [8].

In this paper, three theorems are presented. The first theorem provides sufficient conditions for the asymptotic stability of a second order scalar system. These conditions are more general than the ones provided by the previously mentioned contributions. The second theorem focuses on a slightly smaller class of systems but has the advantage of providing a strict Lyapunov function as well as sufficient conditions for exponential stability. Finally, the third theorem derives sufficient stability conditions in the presence of external disturbances. Interestingly enough, its proof will show how the Variable Gradient Method can be employed to build not only Lyapunov functions but also ISS–Lyapunov functions.

The usefulness of the proposed results will be illustrated via applicative examples.

2. Preliminaries

2.1. Properties of the integral operator

For the readers' convenience, some useful properties of the integral operator are recalled:

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(a) Given a function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(y)\, y > 0, \forall y \in \mathbb{R}\backslash\{0\}$ and a function $\kappa : \mathbb{R} \to \mathbb{R}$ such that $\kappa(y) > 0, \forall y \in \mathbb{R}\backslash\{0\}$, then

$$\int_0^x \kappa(y)\, \phi(y)\, dy > 0, \quad \forall x \in \mathbb{R}\backslash\{0\}. \tag{1}$$

(b) Given a continuously differentiable vector field $g : \mathbb{R}^2 \to \mathbb{R}^2$, if

$$\frac{\partial g_1}{\partial x_2} - \frac{\partial g_2}{\partial x_1} = 0, \tag{2}$$

then g is conservative [9]. As a result, its integral does not depend on the path taken and is

$$\int_0^x g(x)\, dy = \int_0^{x_1} g(y_1, 0)\, dy_1 + \int_0^{x_2} g(x_1, y_2)\, dy_2, \tag{3}$$

for all $x = [x_1\ x_2]^T \in \mathbb{R}^2$.

2.2. Variable Gradient Method

The VGM was first introduced in [8] and is a systematic tool for constructing Lyapunov functions. Consider the second-order nonlinear system

$$\dot{x} = f(x) \tag{4}$$

with $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $f(0) = 0$. The steps of the Variable Gradient Method are the following:

1. Define a family of potential Lyapunov functions $V : \mathbb{R}^2 \to \mathbb{R}$ as

$$\mathrm{V}(x) = \int_0^x g(y)\, dy, \tag{5}$$

where $g : \mathbb{R}^2 \to \mathbb{R}^2$ is a generic conservative vector field.

2. Determine an expression of $g(x)$ which ensures

$$\dot{V} = g(x) \cdot f(x) \le 0, \quad \forall x \ne 0. \tag{6}$$

3. Having obtained a specific expression of $g(x)$, verify that (5) is a positive definite function, i.e. $V(x) > 0, \forall x \in \mathbb{R}^2\backslash\{0\}$.

2.3. Stability notions

Based on [2,10–12], the following definitions are used.

Definition 1. A continuous function $\gamma : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if:

- it is strictly increasing;
- it is such that $\gamma(0) = 0$.

Definition 2. A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if:

- for each fixed s, the function $\beta(r, s)$ is a class-$\mathcal{K}$ function;
- for each fixed r, the function $\beta(r, s)$ is decreasing with respect to s and satisfies $\beta(r, s) \to 0$ for $s \to \infty$.

Definition 3. The origin of a system in the form (4) is

- asymptotically stable if[1] there exists a class-$\mathcal{KL}$ function β and a set $\mathcal{S} \subseteq \mathbb{R}^n$ such that

$$\|x(t)\| \le \beta(\|x(0)\|, t), \quad \forall x(0) \in \mathcal{S}, \forall t \ge 0; \tag{7}$$

- globally asymptotically stable if (7) holds with $\mathcal{S} = \mathbb{R}^n$;

[1] As stated in [2, p. 150], this definition is equivalent to stating that the equilibrium point is stable and attractive.

Definition 4. The origin of a system in the form (4) is

- exponentially stable if there exist positive constants c, k, λ such that

$$\|x(t)\| \le k\, \|x(0)\| \exp(-\lambda t), \quad \forall \|x(0)\| < c. \tag{8}$$

- semi-globally exponentially stable if, for any $c > 0$, there exist positive constants k, λ satisfying (8).
- globally exponentially stable if there exist positive constants k, λ that satisfy (8) $\forall x \in \mathbb{R}^n$;

Definition 5. A system in the form $\dot{x} = f(x, u)$ with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $f(0, 0) = 0$ is

- Input-to-State Stable (ISS) with restriction $\mathcal{X} \subset \mathbb{R}^n$ on the initial state $x(0)$ and restriction $\mathcal{U} \subset \mathbb{R}^m$ on the input $u(t)$ if there exist a class-$\mathcal{KL}$ function β and a class-$\mathcal{K}$ function γ such that

$$\|x(t)\| \le \beta(\|x(0)\|, t) + \gamma\left(\sup_{\tau \le t} \|u(\tau)\|\right) \tag{9}$$

for all $x(0) \in \mathcal{X}$, $u(t) \in \mathcal{U}$.

- ISS with no restrictions (or simply "ISS") if (9) holds with $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{U} = \mathbb{R}^m$.

3. Main results

This section will provide sufficient conditions for characterizing the stability properties of a class of second-order systems. The main interest in these results is that the conditions are very easy to verify. Suitable Lyapunov functions are constructed within the proof of each statement.

Theorem 1. *The origin of the second-order system*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\phi_1(x_1) - \phi_2(x_1)\, x_2^2 - \psi(x_1, x_2) \end{cases} \tag{10}$$

is Globally Asymptotically Stable (GAS) if $\phi_i : \mathbb{R} \to \mathbb{R}$ for $i = 1, 2$ and $\psi : \mathbb{R}^2 \to \mathbb{R}$ are locally Lipschitz continuous functions such that:

$$\phi_i(0) = 0, \quad i = 1, 2$$
$$\phi_1(x_1)\, x_1 > 0, \quad \forall x_1 \in \mathbb{R}\backslash\{0\}$$
$$\phi_2(x_1)\, x_1 \ge 0, \quad \forall x_1 \in \mathbb{R}$$
$$\psi(x_1, x_2)\, x_2 > 0, \quad \forall x_1 \in \mathbb{R}, \forall x_2 \in \mathbb{R}\backslash\{0\}$$

and

$$\int_0^{\pm\infty} \phi_1(y)\, dy = \pm\infty.$$

Proof. The statement will be proven by using the Variable Gradient Method to construct a *weak* Lyapunov function. To this end, define $V(x)$ as in (5) and consider the vector field

$$g(x) = \begin{bmatrix} \alpha(x)\, x_1 & \delta(x)\, x_2 \end{bmatrix} \tag{11}$$

where $\alpha : \mathbb{R}^2 \to \mathbb{R}$ and $\delta : \mathbb{R}^2 \to \mathbb{R}$ are two unknown functions to be determined. Following from (10), the derivative of $V(x)$ is

$$\dot{V}(x) = \alpha(x)\, x_1 x_2 - \delta(x)\left(\phi_1(x_1) + \phi_2(x_1)\, x_2^2\right) x_2 - \delta(x)\, \psi(x_1, x_2)\, x_2. \tag{12}$$

At this point, the objective is to determine $\alpha(x)$, $\delta(x)$ such that

- $g(x)$ is a conservative vector field;
- $\dot{V}(x)$ is a negative semi-definite function.