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Radius of approximate controllability of linear retarded systems under structured perturbations

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ABSTRACT

In this paper we shall deal with the problem of calculation of the radius of approximate controllability in the Banach state space $\mathbb{K}^n \times L_2([-h_k, 0], \mathbb{K}^n)$ for linear retarded systems of the form $\dot{x}(t) = A_0x(t) + A_1x(t-h_1) + \cdots + A_kx(t-h_k) + Bu(t)$. By using multi-valued linear operators we are able to derive computable formulas for this radius when the system's coefficient matrices are subjected to structured perturbations. Some examples are provided to illustrate the obtained results.

1. Introduction

The problems of measuring the robustness of controllability and stabilizability of dynamical control systems have received a good deal of attention in recent years. Different lines of approach can be found for example in [1–6]. However the attention has mainly been devoted to this problem for finite-dimensional systems and very little is known so far for systems in infinite-dimensional spaces.

It is well-known that for systems with infinite-dimensional state spaces the different concepts of exact controllability and approximate controllability can be considered [7,8]. While exact controllability is generally preserved under small perturbations (see, e.g. [9] and references therein) the approximate controllability, unfortunately, can be destroyed by arbitrarily small perturbations of the system parameters. To see this, let us consider the control system (A, b) described by linear differential equation $\dot{x} = Ax + bu$, $x \in X$, $u \in \mathbb{C}$ where $X = l_2$, the Banach space of all square-summable sequences of complex numbers with the standard basis $\{e_i\}$, $i = 1, 2, \ldots$, A is the left shift operator: $Ae_1 = 0$, $Ae_{i+1} = e_i$, $i = 1, 2, \ldots$ and $b \in X$ is the vector with coordinates $1, 1/2, \ldots, 1/i, \ldots$ or equivalently $b = \sum_{i=1}^{\infty} \frac{1}{i} e_i$. Then A is bounded on X, and by a result in [10, (p. 431)] we have $\overline{\text{span}}\{b, Ab, A^2b, \ldots\} = X$. Therefore, the system (A, b) is approximately controllable (see, e.g [8]). Define (A, b_n), $n = 1, 2, \ldots$ with $b_n = \sum_{i=1}^{n} \frac{1}{i} e_i$, then, clearly $\|(A, b_n) - (A, b)\| = \|b - b_n\| \to 0$ as $n \to \infty$ but all systems (A, b_n) are not approximately controllable. Another example is the system in $X = L_2[0, 1]$ described by integro-differential equation of Volterra type

$$\frac{\partial w(t,\xi)}{\partial t} = \int_0^{\xi} w(t,s)ds + v(\xi)u(t), \tag{1.1}$$

with $x(t) = w(t, \cdot) \in L_2[0, 1]$, $t \geq 0$ and $v(\cdot) \in L_2[0, 1]$ such that $v(\xi) \neq 0$ a.e. on $[0, \delta] \subset [0, 1]$ for some $\delta > 0$. Then as shown in [8, Example 3.2.1 and Remark 3.2.1] (1.1) is approximately controllable in X but becomes not such by arbitrarily small perturbations of $v(\xi)$.

Despite of the above discouraging examples, for the class of dynamical systems described by linear retarded equations the approximate controllability in the Banach space $M_2(\mathbb{K}) := \mathbb{K}^n \times L_2([-h_k, 0], \mathbb{K}^n)$, $\mathbb{K} = \mathbb{C}$ (or $\mathbb{R}$), as seen below, is robust against small perturbations of the system matrices. Therefore, it makes sense to consider for such systems the problem of calculation of the radius of approximate controllability which measures the distance from an approximate controllable system to the nearest system which is not approximately controllable.

In this paper we will consider the linear retarded system in $\mathbb{K}^n$, $\mathbb{K} = \mathbb{C}$ (or $\mathbb{R}$) of the form

$$\begin{cases} \dot{x}(t) = A_0x(t) + A_1x(t-h_1) + \cdots + A_kx(t-h_k) + Bu(t), \\ x(0) = x_0, \qquad x(\theta) = \phi_0(\theta), \quad \forall \theta \in [-h_k, 0), \end{cases} \tag{1.2}$$

where $0 = h_0 < h_1 < \cdots < h_k$, $A_i \in \mathbb{K}^{n\times n}$, $i = 0, 1, \ldots, k$, $B \in \mathbb{K}^{n\times m}$, and $\phi_0(\cdot) \in L_2([-h_k, 0], \mathbb{K}^n)$ is a square integrable function.

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System (1.2) is called *approximately controllable* in the Banach space $M_2(\mathbb{K})$, if for any initial state $(x_0, \phi_0(.)) \in M_2(\mathbb{K})$ and desired final state $(x_1, \phi_1(\cdot)) \in M_2(\mathbb{K})$ and arbitrary $\epsilon > 0$, there exists $T > 0$ and a measurable control function $u(\cdot), u(t) \in \mathbb{K}^m$ a.e. $t \in [0, T]$ such that the corresponding solution of (1.2) $x(t) = x(t, x_0, \phi_0, u)$ satisfies

$$\|(x(T), x(T+\cdot)) - (x_1, \phi_1(\cdot))\|_{M_2} = \|x(T) - x_1\|_{\mathbb{K}^n} + \|x(T+\cdot) - \phi_1(\cdot)\|_{L_2} < \epsilon. \tag{1.3}$$

System (1.2) is called *Euclidean controllable* if instead of (1.3), the solution $x(t) = x(t, x_0, \phi_0, u)$ satisfies

$$\|x(T) - x_1\|_{\mathbb{K}^n} = 0. \tag{1.4}$$

Controllability of linear retarded systems was a topic of extensive study in control theory in the eighties of the last century, see e.g. [11–13]. It was shown in particular that (1.2) can be equivalently described by the abstract linear equation in Banach space $M_2(\mathbb{K})$ as $\dot{z} = \mathcal{A}z + \mathcal{B}u$, where $z(t) = (x(t), x(t+\theta)), \theta \in [-h, 0]$, $\mathcal{A}$ is a generator of C_0-semigroup and $\mathcal{B}$ is a compact operator and hence exact controllability never occurs for (1.2) (see e.g. [14]).

The purpose of this paper is to derive formulas for calculating the robustness measure of approximate controllability in Banach space $M_2(\mathbb{K})$ for retarded systems (1.2) when the system matrices A_i, B are subjected to perturbations. Note that the similar problem was considered recently in [15,16] but for the notion of Euclidean controllability. As in [16], we shall apply the approach based on multi-valued linear operators (see e.g. [17,5,18]) to establish some formulas for the distance from an approximate controllable linear retarded system (1.2) to the nearest retarded system which is not approximately controllable when the system matrices are subjected to different classes of perturbations, including particularly separate unstructured perturbations

$$A_i \rightsquigarrow A_i + \Delta_{A_i}, \quad i = 0, 1, \ldots, k, \qquad B \rightsquigarrow B + \Delta_B \tag{1.5}$$

as well as separate perturbations of affine structure

$$A_i \rightsquigarrow A_i + D_i\Delta_{A_i}E_i, \quad i = 0, 1, \ldots, k, \qquad B \rightsquigarrow B + D_B\Delta_B E_B, \tag{1.6}$$

where Δ_{A_i}, Δ_B are unknown disturbances and D_i, E_i, D_B, E_B are given structuring matrices. In some particular cases, the main results yield new computable formulas of complex and real structured controllability radii of linear retarded systems. We also investigate some relationships between the approximate controllability radius and the Euclidean controllability radius.

The organization of the paper is as follows. In the next section we shall present the formulas for complex approximate controllability radii and some relationships with complex Euclidean controllability radius. Section 2 will be devoted to study the real controllability radii under structured perturbations and derive the computable formulas in some special cases. In conclusion we summarize the obtained results and give some remarks of further investigation.

For the readers' convenience, we give a list of notations to be used in what follows. Throughout the paper, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, the field of complex or real numbers, respectively. $\mathbb{K}^{n\times m}$ will stand for the set of all $(n \times m)$-matrices, $\mathbb{K}^n (=\mathbb{K}^{n\times 1})$ is the n-dimensional columns vector space equipped with the vector norm $\|\cdot\|$ and its dual space can be identified with $(\mathbb{K}^n)^* = (\mathbb{K}^{n\times 1})^*$, the rows vector space equipped with the dual norm. For $A \in \mathbb{K}^{n\times m}, A^* \in \mathbb{K}^{m\times n}$ denotes its adjoint matrix and for $A_i \in \mathbb{K}^{n\times m_i}$, $i = 1, 2, \ldots, k$, $[A_1, A_2, \ldots, A_k]$ will denote the $n \times (m_1 + m_2 + \cdots m_k)$-matrix aggregated by columns of A_i. A set-valued map $\mathcal{F} : \mathbb{K}^m \rightrightarrows \mathbb{K}^n$ is said to be multi-valued linear operator if its graph $\operatorname{gr} \mathcal{F} = \{(x, y) : y \in \mathcal{F}(x)\}$ is a linear subspace of $\mathbb{K}^m \times \mathbb{K}^n$. The readers are referred to [5] for the definitions and the properties of multi-valued linear operators which are needed to derive the main results of this paper. In particular, for each multi-valued linear operator $\mathcal{F}$ the adjoint $\mathcal{F}^*$ and the inverse $\mathcal{F}^{-1}$ are well defined as multi-valued linear operators and we have the following useful relations:

$$(\mathcal{F}^*)^{-1} = (\mathcal{F}^{-1})^*, \qquad (\mathcal{G}\mathcal{F})^* = \mathcal{F}^*\mathcal{G}^*, \qquad \|\mathcal{F}\| = \|\mathcal{F}^*\|. \tag{1.7}$$

Here the norm of $\mathcal{F}$ is defined as

$$\|\mathcal{F}\| = \sup\Big\{\inf_{y\in\mathcal{F}(x)} \|y\| : x \in \operatorname{dom} \mathcal{F},\ \|x\| = 1\Big\}. \tag{1.8}$$

If we identify a matrix $F \in \mathbb{K}^{n\times m}$ with a linear operator $F : \mathbb{K}^m \to \mathbb{K}^n$ then its dual operator $F^* : (\mathbb{K}^n)^* \to (\mathbb{K}^m)^*$ is defined by $F^*(y^*) = y^*F$ and its inverse in terms of multi-valued linear operators is defined as $F^{-1}(y) = \{x \in \mathbb{K}^m : Fx = y\}$. Moreover, if F is surjective (i.e. $F(\mathbb{K}^m) = \mathbb{K}^n$) and vector spaces $\mathbb{K}^n, \mathbb{K}^m$ are equipped with Euclidean norms (i.e. $\|x\| = \sqrt{x^*x}$) then the Moore–Penrose pseudo inverse matrix $F^\dagger = F^*(FF^*)^{-1} \in \mathbb{K}^{m\times n}$ exists and defines a linear selector of F^{-1} (i.e. $F^\dagger y \in F^{-1}(y), \forall y \in \mathbb{K}^n$) satisfying $\|F^\dagger y\| = \inf\{\|x\| : x \in F^{-1}(y)\}$ (see [5], Lemma 3.3). This implies, in particular, that

$$\|F^\dagger y\| \le \|x\|, \quad \text{for all } x \in F^{-1}(y). \tag{1.9}$$

2. Complex controllability radius

Consider the linear retarded system (1.2) in $\mathbb{K}^n$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. The characteristic quasi polynomial of system (1.2) is defined as

$$P(\lambda) = A_0 + e^{-h_1\lambda}A_1 + \cdots + e^{-h_k\lambda}A_k - \lambda I_n. \tag{2.1}$$

Then it is well-known (see, e.g. [19,12,13]) that system (1.2) is Euclidean controllable iff

$$\operatorname{rank}[P(\lambda), B] = n \quad \text{for all } \lambda \in \mathbb{C}, \tag{2.2}$$

and is approximately controllable in Banach space $M_2(\mathbb{K})$ iff

$$\text{(i) } \operatorname{rank}[P(\lambda), B] = n \quad \text{for all } \lambda \in \mathbb{C}, \qquad \text{(ii) } \operatorname{rank}[A_k, B] = n. \tag{2.3}$$

It follows in particular that if system (1.2) is approximately controllable in $M_2(\mathbb{K})$ then it is also Euclidean controllable. Moreover, (2.2) and (2.3) imply that controllability of system (1.2) persists under small perturbations of matrices A_i, $i = 0, \ldots, k$ and B.

Now, assume that the matrices of system (1.2) are subjected to structured perturbations of the form

$$[A_0, A_1 \ldots, A_k, B] \rightsquigarrow [\widetilde{A}_0, \widetilde{A}_1, \ldots, \widetilde{A}_k, \widetilde{B}] = [A_0, A_1 \ldots, A_k, B] + D\Delta E, \tag{2.4}$$

so that the perturbed system is described as

$$\dot{x}(t) = \widetilde{A}_0x(t) + \widetilde{A}_1x(t-h_1) + \cdots + \widetilde{A}_kx(t-h_k) + \widetilde{B}u(t). \tag{2.5}$$

Here $\Delta \in \mathbb{K}^{l\times q}$ is unknown disturbance matrix and $D \in \mathbb{K}^{n\times l}$, $E \in \mathbb{K}^{q\times(n(k+1)+m)}$ are given matrices determining the structure of perturbations. Note that the class of affine perturbations $D\Delta E$ has been considered in many papers on control problems in the state space representation and was proved to be very useful in the theory of robust stability and robust controllability (see, e.g. [3,20]). For the sake of brevity, we shall use the notation

$$\underline{A} = [A_0, A_1, \ldots, A_k] \in \mathbb{K}^{n\times(n(k+1))}.$$