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Relaxed conditions for the stability of switched nonlinear triangular systems under arbitrary switching[☆]

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ABSTRACT

This paper addresses the stability of a class of nonlinear switching systems under arbitrary switching. It focuses on switching systems whose modes are made of a cascade of nonlinear scalar systems. It relaxes the stability conditions proposed by Angeli and Liberzon, by relying on the Strong iISS property which is known to be preserved under cascade interconnection. Its applicability is illustrated by the study of switched bilinear systems in triangular form.

1. Introduction

Providing conditions under which a switching system is globally asymptotically stable under arbitrary switching (UGAS) has been the object of intense research. It is well known that switching among nonlinear dynamics that share a common Lyapunov function is UGAS. For linear time-invariant (LTI) systems, UGAS is guaranteed if the state matrices of the individual modes are Hurwitz and either (i) commute with one another, (ii) are symmetric, or (iii) are normal: see [1,2] and references therein for more details. See also [3] for a thorough study of the 2-dimensional case.

A powerful way to study stability of a high-order dynamical system is to see it as the interconnection of simpler dynamical systems. In general, this interconnection is bidirectional, in which case results such as the small-gain theorem have to be used in order to infer the stability of the overall system [4]. However, the complexity of the analysis is dramatically reduced when the subsystems are interconnected in a unidirectional way; in other words, when the overall system is in triangular form [5–8]. It is well known that the switching among LTI systems with Hurwitz triangular matrices is UGAS. This feature has been extended in [9] to nonlinear switched systems. It was shown, in particular, that if the individual modes are all made of a cascade of scalar subsystems, then GAS is preserved under arbitrary switching provided that the subsystems are input-to-state stable (ISS, [10]) with respect to the state of the driving subsystems. The proof of this result relies on the property that ISS is preserved under cascade interconnection [11].

Recently, a less conservative property than ISS has been introduced, which still retains the feature of being preserved under cascade interconnection [12]. This property is known as the Strong iISS, and constitutes an interesting compromise between the generality of integral-ISS (iISS, [13]) and the strength of ISS. Strong iISS was introduced in [14], together with Lyapunov tools to ensure it in practice.

The goal of this paper is therefore to relax the conditions of [9] under which triangular switched systems are UGAS, by requiring that each stage of the triangular system is Strongly iISS rather than ISS. This relaxation extends the class of systems that can be addressed with this technique. It allows for instance to consider specific classes of switching systems whose individual modes are stabilized by saturated feedback [15], and allows to cope with systems whose dynamics switches among bilinear systems.

The main challenge in relaxing the conditions of [9] by exploiting the Strong iISS property stands in the fact that no characterization of Strong iISS involving a single Lyapunov function has yet been established (see [14] for a counter-example on a natural conjecture to that regards). Consequently, we need to rely either on

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solution based reasoning, or on the existing Lyapunov characterizations for ISS [16] and iISS [17].

The paper is organized as follows: after recalling some necessary definitions, we describe the class of nonlinear switched systems considered here and provide our main result (Section 2), we then apply this result to switched bilinear systems (Section 3). All proofs are provided in Section 4.

Notation. Given $\rho \in \mathbb{N}_{\geq 1}$, $\mathcal{P}$ denotes the set of all piecewise constant signals $\sigma : \mathbb{R}_{\geq 0} \to P := \{1, \ldots, \rho\}$ that admit a finite number of discontinuities over any finite time interval. $\mathcal{PD}$ denotes the set of all continuous functions $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\alpha(0) = 0$ and $\alpha(s) > 0$ for all $s > 0$. A class $\mathcal{K}$ function is an increasing $\mathcal{PD}$ function. The class $\mathcal{K}_\infty$ denotes the set of all unbounded $\mathcal{K}$ functions. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for any $t \in \mathbb{R}_{\geq 0}$ and $\beta(s, \cdot)$ is non-increasing and tends to zero as its argument tends to infinity. Given $x \in \mathbb{R}^n$, $|x|$ stands for its Euclidean norm: $|x| := \sqrt{x_1^2 + \cdots + x_n^2}$. Given $\epsilon > 0$, $\mathrm{sat}_\epsilon(s) := \mathrm{sign}(s) \min\{\epsilon, |s|\}$ for all $s \in \mathbb{R}$. Given $m \in \mathbb{N}_{\geq 1}$, $\mathrm{sat}_\epsilon^m(d) := (\mathrm{sat}_\epsilon(d_1), \ldots, \mathrm{sat}_\epsilon(d_m))^T$ for all $d \in \mathbb{R}^m$. $\mathcal{U}^m$ denotes the set of all measurable, locally essentially bounded functions $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$. Given $u \in \mathcal{U}^m$, $\|u\| := \mathrm{ess\,sup}_{t \geq 0}|u(t)|$.

2. Problem statement and main result

2.1. Preliminary definitions

The present paper studies the stability of a particular class of switched nonlinear systems. Generally speaking, a switched nonlinear system is defined as

$$\dot{x} = f_\sigma(x)$$

where $\sigma \in \mathcal{P}$ and, for each $p \in P$, $f_p : \mathbb{R}^n \to \mathbb{R}^n$ denotes a locally Lipschitz function. This system is said to be *uniformly globally asymptotically stable* (UGAS) if there exists a $\mathcal{KL}$ function β such that, for all $x^0 \in \mathbb{R}^n$ and all $\sigma \in \mathcal{P}$, its solution satisfies $|x(t; x^0, \sigma)| \leq \beta(|x^0|, t)$ for all $t \geq 0$.

Similarly to [9], this paper exploits specific input-to-state properties of the individual subsystems to derive stability of the overall switched system. This requires us to consider switched systems with inputs:

$$\dot{x} = f_\sigma(x, u) \tag{1}$$

where $u \in \mathcal{U}^m$ represents the exogenous input. This system is said to be *uniformly input-to-state stable with respect to small inputs* (ISS wrt small inputs) if there exist a positive constant R, called *input threshold*, $\beta \in \mathcal{KL}$, and $\gamma \in \mathcal{K}_\infty$ such that, for all $x^0 \in \mathbb{R}^n$, all $\sigma \in \mathcal{P}$, and all $u \in \mathcal{U}^m$, its solution satisfies

$$\|u\| \leq R \Rightarrow |x(t; x^0, \sigma, u)| \leq \beta(|x^0|, t) + \gamma(\|u\|), \quad \forall t \geq 0.$$

If the right-hand side of the above implication holds for all $u \in \mathcal{U}^m$, we recover the notion of input-to-state stability (ISS) introduced in [18] and widely studied in the literature: see [10] for a survey. ISS wrt small inputs thus requires ISS as long as the input amplitude does not overpass the input threshold R. It should not be confused with the notion of local ISS (LISS), used for instance in [19], for which the above estimate is required to hold only for small initial conditions and small input amplitudes: on the contrary, ISS wrt small inputs constrains the input magnitude but not the initial state. Note that ISS wrt to small inputs implies UGAS of the switched system in the absence of exogenous inputs ($u = 0$). It also guarantees a bounded response to any input whose amplitude is below the input threshold R. Nonetheless, it provides no information on the behavior of the system when the applied input has an amplitude greater than R: solutions may diverge or even escape in finite time. To prevent this, the Strong iISS property introduced in [14] combines it with integral input-to-state stability (iISS, [13]). More precisely, we say that (1) is *Strongly iISS* if it is both ISS wrt to small inputs and iISS, the latter meaning that there exist $\beta \in \mathcal{KL}$ and $\eta_1, \eta_2 \in \mathcal{K}_\infty$ such that, for all $x^0 \in \mathbb{R}^n$, all $u \in \mathcal{U}^m$, and all $\sigma \in \mathcal{P}$,

$$|x(t; x^0, \sigma, u)| \leq \beta(|x^0|, t) + \eta_1\left(\int_0^t \eta_2(|u(s)|)ds\right), \quad \forall t \geq 0.$$

By combining iISS and ISS wrt small inputs, the Strong iISS property thus ensures existence of solutions at all times, bounded state in response to any input signal whose amplitude is below the input threshold R, and a vanishing state in response to any vanishing input: see [14] for further details.

We stress that the extension of these properties to non-switched systems can be straightforwardly deduced by the above definitions by considering that σ can only take a single value (namely, $P = \{1\}$).

2.2. Problem statement

While Strong iISS a much weaker robustness property than ISS, they share a crucial common feature: they are both preserved under cascade interconnection [14,11]. This feature was extensively exploited in [9] to analyze the stability of switched nonlinear systems in triangular form. The objective of this paper is to generalize this result, by replacing the ISS requirement by the more general Strong iISS property.

Let us start by formally introducing the class of systems considered here. Given $n \in \mathbb{N}_{\geq 1}$ we address the following switched dynamics:

$$\begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_i \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} f_\sigma^1(x_1, x_{2\to n}) \\ \vdots \\ f_\sigma^i(x_i, x_{i+1\to n}) \\ \vdots \\ f_\sigma^{n-1}(x_{n-1}, x_n) \\ f_\sigma^n(x_n) \end{pmatrix}, \tag{2}$$

where $\sigma \in \mathcal{P}$, $x_i \in \mathbb{R}$, and $x_{i\to j} := \left(x_i, x_{i+1}, \ldots, x_j\right)^T \in \mathbb{R}^{j-i+1}$ for all $1 \leq i \leq j \leq n$. We stress that the dynamics of each state variable x_j is only influenced by its own value and those of the driving states $x_{j+1\to n}$; for each given value of the signal σ, the right-hand side of (2) is thus given by a cascade of scalar subsystems: we refer to such systems as nonlinear triangular switched systems. We make the following two assumptions, on the individual subsystems.

Assumption 1. For each $p \in P$, the origin of the driving system $\dot{x}_n = f_p^n(x_n)$ is globally asymptotically stable (GAS).

Assumption 2. For each $i \in \{1, \ldots, n-1\}$ and each $p \in P$ the system $\dot{x}_i = f_p^i(x_i, x_{i+1\to n})$ is Strongly iISS with respect to the input $x_{i+1\to n}$.

Note that the above assumptions concern only non-switched dynamics.

2.3. Main result

We are now ready to state our main result.

Theorem 1. *Under Assumptions 1 and 2, the switched system (2) is uniformly globally asymptotically stable (UGAS).*

This result is similar to [9, Theorem 1] as it provides a condition under which specific switched nonlinear systems in triangular form are UGAS. The main difference stands in the main assumption