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# Distributed gradient algorithm for constrained optimization with application to load sharing in power systems[☆]

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ABSTRACT

In this paper, a distributed constrained optimization problem is discussed to achieve the optimal point of the sum of agents' local objective functions while satisfying local constraints. Here neither the local objective function nor local constraint functions of each agent can be shared with other agents. To solve the problem, a novel distributed continuous-time algorithm is proposed by using the KKT condition combined with the Lagrangian multiplier method, and the convergence is proved with the help of Lyapunov functions and an invariance principle for hybrid systems. Furthermore, this distributed algorithm is applied to optimal load sharing control problem in power systems. Both theoretical and numerical results show that the optimal load sharing can be achieved within both generation and delivering constraints in a distributed way.



## 1. Introduction

Recent years have witnessed increasing research attention on distributed optimization and its applications in many engineering systems (see [1–10]). In fact, distributed optimization algorithms with a global objective function as the sum of agents' local objective functions have been proposed in the multi-agent systems where each agent's local objective/constraint function cannot be known or shared by other agents, maybe due to the privacy concern, computational burden, or communication cost/failure.

Actually, optimization problems often involve certain constraints, and great efforts keep devoted to solve the constrained optimization problems in a distributed way (see [2–8]). Projection-based distributed algorithm was proposed in [2], and further investigated in [5] and [7] for set constrained optimization. Lagrangian multiplier method was investigated in [4], while a penalty-based method was proposed in [8], both for function constrained problems. Meanwhile, dual decomposition was applied to separable problems with affine constraints in [3,6]. However, those results (with discrete-time algorithms) mainly addressed the problems where all the agents have the same constraints, which may be restrictive in some situations.

The continuous-time dynamics for distributed optimization attract more and more attention by taking advantage of the well-developed continuous-time control techniques and by concerning the implementation of physical systems (see [11–17]). The continuous-time optimization algorithm was studied in the seminal work [18] and then investigated with various backgrounds (referring to [19,20]). Recently, there have been discussions on continuous-time distributed optimization. For example, a continuous-time dynamics was proposed to show connectivity conditions for the convex set intersection computation in [12]. A second-order distributed dynamics was proposed to solve an unconstrained optimization in [11], while a similar algorithm was also constructed with non-smooth objective functions in [16]. Moreover, a distributed optimization algorithm based on proportional–integral control was given in [15], and later internal model principle was employed to achieve exact optimization with capability of rejecting external disturbance in [13]. However, to our knowledge, very few continuous-time distributed algorithms for constrained optimization have ever been documented.

The distributed load sharing optimization problem has been widely investigated in power systems (see [21–24]). It aims to find the optimal generation allocation to share the loads within both the generation and the transmission capacity bounds, which can be formulated as a class of distributed constrained optimization

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problems. [21] provided an interesting insight that the frequency and power flow dynamics are closely related with a primal–dual optimization algorithm. [22–24] considered the distributed load sharing control with or without generation capacity constraints. However, such works have not taken the transmission line capacity constraints into consideration, which is crucial for the practicality of the load sharing control. Theoretically, the distributed load sharing optimization considering both the generation and the transmission constraints can lead to a new class of distributed constrained optimization problem which is the focus of this paper.

In this paper, we study a distributed optimization problem for agents with their local inequality constraints, and its application to the load sharing optimization. Different from the existing results such as those given in [2–8] and [25,11–17], our distributed algorithm enables the agents to find the optimal point with respect to the sum of the local objective functions while satisfying all the local constraints. Note that the optimal solution must be within the intersection set of each agent's local private feasible set specified by the local constraints, while neither local objective function nor local constraint functions of each agent can be known or shared by other agents.

To solve the complicated optimization problem, we propose a novel continuous-time distributed algorithm by using the KKT condition and saddle point property, and analyze the algorithm with constructed Lyapunov functions and hybrid LaSalle invariance principle. Then we apply the proposed algorithm to the optimal load sharing problem considering both the generation limits of the power generators and the delivering limits of the transmission lines. This extends the existing results presented in [21–24], leading to a more practical optimal distributed load sharing control. We also provide the simulation experiments to show the effectiveness of the proposed method.

The paper is organized as follows. The distributed constrained optimization problem is formulated in Section 2, while the continuous-time algorithm is proposed in Section 3. Then the convergence of the algorithm is proved along with a numerical experiment in Section 4. Moreover, the application to the distributed load sharing optimization in power systems is shown in Section 5. Finally, the concluding remarks are given in Section 6.

*Notations*: Denote $\mathbf{1}_m = (1, \dots, 1)^T \in \mathbf{R}^m$ and $\mathbf{0}_m = (0, \dots, 0)^T \in \mathbf{R}^m$. For a column vector $x \in \mathbf{R}^m$, $x^T$ denotes its transpose. $I_n$ denotes the identity matrix in $\mathbf{R}^{n\times n}$. For a matrix $A = [a_{ij}]$, $a_{ij}$ or $A_{ij}$ stands for the matrix entry in the $i$th row and $j$th column of $A$.

## 2. Problem formulation

In this section, we give the formulation of a distributed optimization problem with local inequality constraints.

Consider a group of agents, $\mathcal{N} = \{1, \dots, N\}$, where each has **local objective function** $f_i(x)$ and **local inequality constraints** $g_j^i(x) \le 0, j = 1, \dots, J^i$. The agents need to optimize the sum of their local objective functions $f_i(x)$ under all the agents' local constraints. Since both the local objective functions $f_i(x)$ and local constraints $g_j^i(x), j = 1, \dots, J^i$ are only known by agent $i$ and cannot be shared with other agents, the optimization has to be achieved with the cooperation of all the agents in a distributed way. To be strict, we consider

**Problem 1.**

$$\min f(x), \quad f(x) = \sum_{i=1}^{N} f_i(x)$$
$$\text{subject to} \quad g_j^i(x) \le 0, \quad j = 1, \dots, J^i, i = 1, \dots, N,$$

where $x \in \mathbf{R}^m$ is the decision variable, and $f_i(x), g_j^i(x), j = 1, \dots, J^i, i = 1, \dots, N$ are twice continuously differentiable and convex functions over $\mathbf{R}^m$, which are only known by agent $i$.

**Remark 2.1.** Problem 1 is different from the formulations given in [2,4,5,7,8], because each agent has local private convex feasible set $X_i = \{x \in \mathbf{R}^m | g_j^i(x) \le 0, j = 1, \dots, J^i\}$ specified by local constraint functions. Moreover, Problem 1 is not the separable one considered in [3,6] and [25], because the decision variable $x$ is common for all the agents. Therefore, the agents need to find one common point within the intersection set $X = \bigcap_{i=1}^N X_i$ in order to minimize the sum of local objective functions, without knowing other agents' feasible sets.

Then we give the following assumptions for Problem 1:

**Assumption 1.** At least one of the local objective functions has positive definite Hessian $\nabla^2 f_i(x)$ over $x \in \mathbf{R}^m$.

Assumption 1 implies that at least one of the local objective functions is *strictly* convex, hence the uniqueness of the optimal solution (see Theorem 2.69 in [26]).

**Assumption 2.** Problem 1 has finite optimal solution, and $X = \bigcap_{i=1}^N X_i$ has nonempty interior point.

Assumption 2 guarantees Slater's constraint qualification condition, and moreover, this assumption implies that there exists finite $x^*$ such that $x^* = \arg\min f(x) = \sum_{i=1}^N f_i(x), x \in X$, and there exists at least one interior point $x_0$ of $X$ with $g_j^i(x_0) < 0, j = 1, \dots, J^i, i = 1, \dots, N$.

From Theorems 3.25, 3.26 and 3.27 in [26], we have the following result.

**Lemma 2.2.** *With Assumptions 1 and 2, the point $x^*$ is the optimal solution of Problem 1 if and only if there exist Lagrangian multipliers $\lambda_{ij}^* \ge 0, j = 1, \dots, J^i, i = 1, \dots, N$ (or denoted as $\{\lambda_{ij}^*\}$) satisfying the following KKT condition.*

$$\begin{aligned} &\sum_{i=1}^{N} \nabla f_i(x^*) + \sum_{i=1}^{N}\sum_{j=1}^{J^i} \lambda_{ij}^* \nabla g_j^i(x^*) = \mathbf{0} \\ &g_j^i(x^*) \le 0, \qquad \lambda_{ij}^* g_j^i(x^*) = 0, \quad j = 1, \dots, J^i,\ i = 1, \dots, N. \end{aligned} \tag{1}$$

*Furthermore, the set of multipliers $\{\lambda_{ij}^*\}$ satisfying KKT condition (1) is closed, convex, and bounded.*

By Lemma 2.2, the distributed optimization task is to cooperatively find the point $x^*$ with multipliers $\{\lambda_{ij}^*\}$ to satisfy (1).

In the multi-agent network, each agent can exchange information only with some neighbor agents. The network topology can be described by a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with $\mathcal{N} = \{1, \dots, N\}$ representing the agents set and $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ containing all the information interactions between agents. If agent $i$ can get information from agent $j$, then $(j, i) \in \mathcal{E}$. The graph $\mathcal{G}$ is undirected when $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$. A path of graph $\mathcal{G}$ is a sequence of distinct agents in $\mathcal{N}$ such that any consecutive agents in the sequence corresponding to an edge of the graph $\mathcal{G}$. Agent $j$ is said to be connected to agent $i$ if there is a path from $j$ to $i$. Graph $\mathcal{G}$ is said to be connected if any two agents are connected. Define the adjacency matrix $A = [a_{ij}]$ associated with $\mathcal{G}$ with $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Then the Laplacian of graph $\mathcal{G}$ is $L = Deg - A$ with the degree matrix $Deg = diag\{\sum_{j=1}^N a_{1j}, \dots, \sum_{j=1}^N a_{Nj}\}$. More details about graph theory for multi-agent network can be found in [27]. The following assumption is about the connectivity of graph $\mathcal{G}$, which guarantees that any agent's information can reach any other agents.