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# Convergence domain for time-varying switched systems

Jinghui Suo, Jitao Sun *

*Department of Mathematics, Tongji University, Shanghai 200092, PR China*

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ABSTRACT

In this paper, we investigate the convergence domain for time-varying switched systems. We construct multiple Lyapunov functions and each Lyapunov function is decreasing outside a ball whose radius is time-varying. We give a relatively accurate convergence domain for a general time-varying switched system. Finally, a numerical example, a mass–spring–damper system, is provided to show the effectiveness of the theoretical results.



## 1. Introduction

Stability is an important issue for dynamical systems. However sometimes there is no need to achieve stability in practice. In certain engineering applications, it is more natural to identify whether the solutions of dynamical systems are bounded, see [1–9] and references therein.

Lyapunov analysis is very useful to show boundedness of the solutions. When we observe the derivative of a Lyapunov function along the solution of a differential dynamical system, it is common that this derivative is not always negative but is indeed negative when the state trajectory satisfies some conditions. [2,3,5,10] considered the case that the derivative of a Lyapunov function was negative when the norm of the state was greater than or equal to a constant and provided a theorem about ultimate boundedness of the solution. Recently, Zhao et al. [4] investigated a more general case that the derivative of a Lyapunov function was negative when the norm of the state was greater than or equal to a certain function of time and gave a convergence domain for a general class of time-varying nonlinear systems.

A switched system is a dynamical system consisting of a family of subsystems and a switching signal that determines the switching between them. In recent years, switched systems have gained considerable attention in science and engineering and a number of criteria have been derived, see [7–19] and references therein. In view of the interaction between continuous dynamics and discrete dynamics, switched systems may have very complicated behaviors. For instance, switching may destabilize a switched system even if all individual subsystems are stable, whereas suitably constrained switching between unstable subsystems may give rise to stability. Motivated by this, some scientists have focused on the boundedness of the solutions of switched systems [7–10]. However, these researches do not provide the relatively accurate convergence domain for a general switched system. In this paper, we study the convergence domain for time-varying switched systems. Since a common Lyapunov function may not exist for a switched system, we use multiple Lyapunov function method and assume that for each Lyapunov function, its derivative is negative when the norm of the state is no less than a function of time. To the best of our knowledge, there has been no result of such research. Therefore the study in this aspect is meaningful and challenging.

This paper is organized as follows. In Section 2, we give the main results about convergence domain for a time-varying switched system. An application example, a mass–spring–damper system, is provided to illustrate the theoretical results in Section 3, following by conclusions in Section 4.

*Notation*: $\|\cdot\|$ refers to the Euclidean norm for vectors. $I$ represents the identity matrix. $B(\bar{x}, r) = \{x \mid \|x - \bar{x}\| \le r\}$ denotes a closed ball with radius $r > 0$ centered at a point $\bar{x}$. A function $\beta$ is said to be of class $\mathcal{K}$ if it is a continuous, strictly increasing function satisfying $\beta(0) = 0$.

## 2. Main results

We consider the following switched system

$$\dot{x} = f_{\sigma(t)}(t, x), \tag{1}$$

where $x \in \mathbb{R}^n$, $\sigma : [0, \infty) \to \mathcal{P} = \{1, 2, \ldots, m\}$ is a switching signal, $m$ is the number of subsystems. $\sigma$ is a piecewise constant

* Corresponding author. Tel.: +86 21 65983241x1307; fax: +86 21 65981985.
*E-mail address:* sunjt@sh163.net (J. Sun).

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function of time such that $\sigma(t)=\sigma(t_k)$, for $t\in[t_k,t_{k+1}),\ k\in\mathbb{N}$. Time sequence $\{t_k\}_{k=0}^{\infty}$ satisfies $0=t_0<t_1<\cdots<t_k<\cdots$ and $t_k\to\infty$ as $k\to\infty$, $\inf_{k\geq0}(t_{k+1}-t_k)\geq\tau>0$, $t_k$ denotes the moment of the $k$-th switching. For each $p\in\mathcal{P}$, $f_p:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous. Here we assume that the solution of system (1) has global existence and is unique. In addition, we assume that the state of the system (1) does not jump at the switching instants, i.e., the solution $x(t)$ is everywhere continuous.

**Assumption 1.** For any $M>0,\ p\in\mathcal{P}$, there exists $H_p\triangleq H_p(M)$, which may be dependent on $M$, such that

$$\|f_p(t,x)\|\leq H_p(M),\quad \forall t\geq0 \text{ and } \|x\|\leq M. \tag{2}$$

**Theorem 1.** *Suppose that Assumption 1 holds. For each $p\in\mathcal{P}$, there exist nonnegative bounded functions $g_p(t)$ defined on $[0,\infty)$, smooth functions $V_p(t,x)$: $[0,\infty)\times\mathbb{R}^n\to[0,\infty)$, class $\mathcal{K}$ functions $\alpha_p,\beta_p,\gamma_p$, positive constants $\mu\geq1,\ \tau>0$ satisfying*

- (A1) *$\alpha_p(\|x\|)\leq V_p(t,x)\leq\beta_p(\|x\|)$, when $\|x\|\geq g_p(t)$, $x\in\mathbb{R}^n$, $t\geq0$.*
- (A2) *$\dot V_p(t,x)\leq-\gamma_p(\|x\|)$, when $\|x\|\geq g_p(t)$.*
- (A3) *$V_p(t,x)\leq\mu V_q(t,x)$, when $\forall p,q\in\mathcal{P}$, $\|x\|\geq\max\{g_p(t),g_q(t)\}$.*
- (A4) *Set $d=\max_{1\leq p\leq m}\alpha_p(\overline{\lim}_{t\to\infty}g_p(t))$, $\eta_p(u)\triangleq\gamma_p(\beta_p^{-1}(u))$, where $\beta_p^{-1}$ is the inverse function of $\beta_p$. There exists a sequence $\{\theta_k\}$ such that for $z>d$, $-(t_{k+1}-t_k)+\int_z^{\mu z}\frac{ds}{\eta_{\sigma(t_k)}(s)}\leq-\theta_k$, $\forall k\in\mathbb{N}$, where $\theta_k\geq0$, $\underline{\lim}_{k\to\infty}\theta_k>0$.*
- (A5) *$V_p(t,x)$ as functions of $x$, are equicontinuous on any bounded region of $\mathbb{R}^n$, uniformly with respect to $t$.*

*Then the solution $x(t)$ of (1) converges to the set*

$$Q=\bigcap_{T\geq b>0}\Big(\bigcup_{a\geq T}Q_{a,b}\Big), \tag{3}$$

*where*

$$\Omega=\left\{x\in\mathbb{R}^n|\ \|x\|\leq\max_{1\leq p\leq m}\overline{\lim_{t\to\infty}}\,g_p(t)\right\}, \tag{4}$$

$$Q_{a,b}=\left\{x|V_{\sigma(a)}(a,x)\leq\mu\sup_{y\in\Omega,s\geq b}\{V_{\sigma(s)}(s,y)\}\right\}. \tag{5}$$

**Proof.** We consider two cases: $\max_{1\leq p\leq m}\overline{\lim}_{t\to\infty}g_p(t)=0$ and $\max_{1\leq p\leq m}\overline{\lim}_{t\to\infty}g_p(t)>0$.

Case 1: $\max_{1\leq p\leq m}\overline{\lim}_{t\to\infty}g_p(t)=0$, that is, $\forall p\in\mathcal{P}$, $\overline{\lim}_{t\to\infty}g_p(t)=0$. Since $g_p(t)\geq0$, it is equivalent to $\lim_{t\to\infty}g_p(t)=0$. In this case, we will show $\lim_{t\to\infty}x(t)=0$ by two steps:

First, we prove that the origin is an accumulation point of $\{x(t)\}$. If this is not true, then for large enough $t$, there exists a $\lambda>0$ such that $\|x\|>\lambda\geq g_p(t)$, $\forall p\in\mathcal{P}$. Let $\varrho=\min\{\alpha_1(\lambda),\ldots,\alpha_m(\lambda)\}$. According to the condition (A1), we have $\varrho\leq\alpha_p(\lambda)\leq\alpha_p(\|x\|)\leq V_p(t,x)\leq\beta_p(\|x\|)$, $\forall p\in\mathcal{P}$. From the condition (A2), we obtain $\dot V_{\sigma(t)}(t,x)\leq-\gamma_{\sigma(t)}(\|x\|)$ for large enough $t$. In view of the fact that $\beta_p$ are class $\mathcal{K}$ functions, $\forall p\in\mathcal{P}$, the inverse functions $\beta_p^{-1}$ also belong to class $\mathcal{K}$. It follows that $\eta_p$ are class $\mathcal{K}$ functions. From the condition (A1), we have $\|x\|\geq\beta_p^{-1}(V_p(t,x))$, $\forall p\in\mathcal{P}$, which yields for large enough $t$,

$$\dot V_{\sigma(t)}(t,x)\leq-\gamma_{\sigma(t)}(\|x\|)\leq-\eta_{\sigma(t)}(V_{\sigma(t)}(t,x)). \tag{6}$$

Now we study $V_{\sigma(t)}(t,x)$ on $t\in[t_i,\infty)$ for large enough switching moment $t_i$. Let $V_{\sigma(t_k)}(t_{k+1}^-,\ x(t_{k+1}^-))\triangleq\lim_{t\to t_{k+1}^-}V_{\sigma(t_k)}(t,x(t))$. For the sake of ease, setting $V_{\sigma(t)}(t)=V_{\sigma(t)}(t,x(t))$ and according to (6), when $t\in[t_{i+k},t_{i+k+1}),\ \forall k\in\mathbb{N}$, we have

$$\int_{t_{i+k}}^{t_{i+k+1}^-}\frac{\dot V_{\sigma(t_{i+k})}(s)ds}{\eta_{\sigma(t_{i+k})}(V_{\sigma(t_{i+k})}(s))}=\int_{V_{\sigma(t_{i+k})}(t_{i+k})}^{V_{\sigma(t_{i+k})}(t_{i+k+1}^-)}\frac{ds}{\eta_{\sigma(t_{i+k})}(s)}\leq t_{i+k}-t_{i+k+1}. \tag{7}$$

In view of the condition (A3), we get

$$\int_{V_{\sigma(t_{i+k})}(t_{i+k+1}^-)}^{V_{\sigma(t_{i+k+1})}(t_{i+k+1})}\frac{ds}{\eta_{\sigma(t_{i+k})}(s)}\leq\int_{V_{\sigma(t_{i+k})}(t_{i+k+1}^-)}^{\mu V_{\sigma(t_{i+k})}(t_{i+k+1}^-)}\frac{ds}{\eta_{\sigma(t_{i+k})}(s)}. \tag{8}$$

Combining (7) and (8), from the condition (A4), we obtain that

$$\int_{V_{\sigma(t_{i+k})}(t_{i+k})}^{V_{\sigma(t_{i+k+1})}(t_{i+k+1})}\frac{ds}{\eta_{\sigma(t_{i+k})}(s)}\leq t_{i+k}-t_{i+k+1}+\int_{V_{\sigma(t_{i+k})}(t_{i+k+1}^-)}^{\mu V_{\sigma(t_{i+k})}(t_{i+k+1}^-)}\frac{ds}{\eta_{\sigma(t_{i+k})}(s)}\leq-\theta_{i+k}.$$

Since $\eta_{\sigma(t_{i+k})}(s)>0$ for $s>0$, we have $V_{\sigma(t_{i+k+1})}(t_{i+k+1})\leq V_{\sigma(t_{i+k})}(t_{i+k})$. Hence by mathematical induction we conclude that

$$V_{\sigma(t_{i+k+1})}(t_{i+k+1})\leq V_{\sigma(t_{i+k})}(t_{i+k})\leq\cdots\leq V_{\sigma(t_i)}(t_i). \tag{9}$$

Furthermore, we have

$$\theta_{i+k}\leq\int_{V_{\sigma(t_{i+k+1})}(t_{i+k+1})}^{V_{\sigma(t_{i+k})}(t_{i+k})}\frac{ds}{\eta_{\sigma(t_{i+k})}(s)}\leq\frac{V_{\sigma(t_{i+k})}(t_{i+k})-V_{\sigma(t_{i+k+1})}(t_{i+k+1})}{\eta(\varrho)}, \tag{10}$$

where $\eta(\varrho)=\min\{\eta_1(\varrho),\ldots,\eta_m(\varrho)\}$. It follows that

$$V_{\sigma(t_{i+k+1})}(t_{i+k+1})\leq V_{\sigma(t_{i+k})}(t_{i+k})-\eta(\varrho)\theta_{i+k}\leq V_{\sigma(t_i)}(t_i)-\eta(\varrho)\sum_{j=i}^{i+k}\theta_j.$$

In view of the fact that $\theta_k\geq0$, $\underline{\lim}_{k\to}\theta_k>0$, then we get $\sum_{j=i}^{\infty}\theta_j=\infty$. Thus $\lim_{k\to\infty}V_{\sigma(t_{i+k+1})}(t_{i+k+1})=-\infty$, which is a contradiction. Therefore the origin is an accumulation point of $\{x(t)\}$.

Then, we show that the origin is the unique accumulation point of $\{x(t)\}$. If it is incorrect, then there exists another accumulation point $x^*\neq0$. Choose constants $c_1>0$, $c_2>0$ such that

$$B(0,c_1)\cap B(x^*,c_2)=\emptyset. \tag{11}$$

On account of the fact that both $x^*$ and the origin are accumulation points, there exists a sequence $\{\tau_k\}$, $\tau_k\to\infty$ such that when $k$ is odd, $\|x(\tau_k)\|=c_1$, for $t\in(\tau_k,\tau_{k+1})$, $\|x(t)\|>c_1$ and when $k$ is even, $\|x(\tau_k)-x^*\|=c_2$. From (11), we have $\|x^*\|>c_2$.

- If $\tau_k$ and $\tau_{k+1}$ are in the same interval between two consecutive switching moments, that is, there exists $r\in\mathbb{N}$ such that $t_r\leq\tau_k<\tau_{k+1}<t_{r+1}$, then when $k$ is odd and large enough, we obtain $\|x(t)\|\geq g_{\sigma(t)}(t)$ and $V_{\sigma(t_r)}(t,x(t))$ is decreasing on $[\tau_k,\tau_{k+1}]$. Since $c_2=\|x(\tau_{k+1})-x^*\|\geq\|x^*\|-\|x(\tau_{k+1})\|$, we obtain $\|x^*\|-c_2\leq\|x(\tau_{k+1})\|$. In view of the condition (A1), we have

$$\begin{aligned}\alpha_{\sigma(t_r)}(\|x^*\|-c_2)&\leq\alpha_{\sigma(t_r)}(\|x(\tau_{k+1})\|)\\&\leq V_{\sigma(t_r)}(\tau_{k+1},x(\tau_{k+1}))\\&<V_{\sigma(t_r)}(\tau_k,x(\tau_k))\leq\beta_{\sigma(t_r)}(\|x(\tau_k)\|)\\&=\beta_{\sigma(t_r)}(c_1).\end{aligned} \tag{12}$$

  Select small enough $c_1$ and obtain $\alpha_{\sigma(t_r)}(\|x^*\|-c_2)>\beta_{\sigma(t_r)}(c_1)$, which contradicts (12).