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A Lyapunov-based small-gain theorem for interconnected switched systems[☆]

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ABSTRACT

Stability of an interconnected system consisting of two switched systems is investigated in the scenario where in both switched systems there may exist some subsystems that are not input-to-state stable (non-ISS). We show that, providing the switching signals neither switch too frequently nor activate non-ISS subsystems for too long, a small-gain theorem can be used to conclude global asymptotic stability (GAS) of the interconnected system. For each switched system, with the constraints on the switching signal being modeled by an auxiliary timer, a correspondent hybrid system is defined to enable the construction of a hybrid ISS Lyapunov function. Apart from justifying the ISS property of their corresponding switched systems, these hybrid ISS Lyapunov functions are then combined to establish a Lyapunov-type small-gain condition which guarantees that the interconnected system is globally asymptotically stable.

1. Introduction

The study of interconnected systems plays a significant role in the development of stability theory of dynamic systems, as it allows one to investigate the stability property of a complex system by analyzing its less complicated components. In this context, the small-gain theorems have proved to be important tools in the analysis of feedback connections of multiple systems, which appear frequently in the control literature. A comprehensive summarization of classical small-gain theorems involving input–output gains of linear systems can be found in [1]. This technique was then generalized to nonlinear feedback systems in [2,3] within the input–output context. The notion of input-to-state stability (ISS) proposed by Sontag [4] was naturally adopted and extended in [5] to establish a general nonlinear small-gain theorem which guaranteed both external and internal stabilities. Instead of analyzing the behavior of solution trajectories, Jiang et al. [6] have developed a Lyapunov-type nonlinear small-gain theorem based on the construction of ISS Lyapunov functions. A variety of nonlinear small-gain theorems were summarized in [7, Section 10.6].

In this paper, we explore the stability property of interconnected nonlinear switched systems. The study of switched systems has attracted a lot of attention in recent years (see, e.g., [8] and references therein). It is well-known that, in general, a switched system does not necessarily inherit the stability properties of its subsystems. For example, in [8, Part II] it is shown that a switched system consisting of two asymptotically stable subsystems may not be stable. In the linear system context, it was proved in [9] that such a switched system can achieve asymptotic stability providing the switching signal satisfies a certain dwell-time condition. This approach was then generalized to the nonlinear system context and to the concept of average dwell-time condition in [10]. In [11] a similar result was developed for a linear switched system with both stable and unstable subsystems by restricting the fraction of time in which the unstable subsystems are active. The study of stability property inheritance in switched systems was extended to the ISS context by Vu et al. [12], and to the IOSS (input/output-to-state stability) context by Müller and Liberzon [13], both for nonlinear switched systems. Furthermore, in [13] the IOSS property of a nonlinear switched system was studied also for the general case where some of the subsystems are not input/output-to-state stable. In [14] a small-gain theorem was formulated to establish the ISS property of a switched interconnected nonlinear system under an average dwell-time condition, and the global stabilization of a switched nonlinear system in strict-feedback form with possibly non-ISS subsystems was investigated based on this small-gain theorem.

In this work, a sufficient condition is formulated to guarantee the global asymptotic stability (GAS) of an interconnected system consisting of two nonlinear switched systems. We have considered

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a very general scenario: in both switched systems there may exist some subsystems that are not input-to-state stable (non-ISS). It is proved that, providing the switching signals neither switch too frequently (average dwell-time constraint) nor activate non-ISS subsystems for too long (time-ratio constraint), a small-gain theorem can be established by introducing auxiliary timers and adopting hybrid system techniques. In particular, for each switched system, a hybrid system is defined such that their solutions are correspondent and the constraints on the switching signal are modeled by the auxiliary timer. For each hybrid system, an ISS Lyapunov function is then constructed to establish the ISS property for all complete solutions to the hybrid system, and therefore all solutions to the switched system. (Although the result that a switched system with not necessarily ISS subsystems is ISS under certain average dwell-time condition and time-ratio condition has already been proved in [13], the Lyapunov-type formulation in this paper exhibits an improvement: it not only generates an ISS Lyapunov function which is used later in the study of the interconnected system, but provides means for robustness analysis as well.) With these two ISS Lyapunov functions, a small-gain condition is then established to prove the GAS property of the interconnected switched system.

Hybrid systems are dynamic systems that possess both continuous-time and discrete-time features. Trajectory-based small-gain theorems for interconnected hybrid systems were first presented in [15,16], while Lyapunov-based formulations were introduced in [17]. The concept of ISS Lyapunov function was extended to hybrid systems in [18]. In our analysis of hybrid systems, we have adopted the modeling framework proposed by Goebel et al. [19], which proved to be general and natural from the viewpoint of Lyapunov stability theory. In the hybrid system context, a detailed study of small-gain theorems based on the construction of ISS Lyapunov functions using this modeling framework can be found in [20–22]. Comparing to [22], our result on modifying the ISS Lyapunov function to guarantee its decrease along solutions is more general in the sense that it applies to the situation where the original ISS Lyapunov functions are increasing both at the jumps and during some of the flows. Based on the idea of restricting non-ISS subsystems' total activation time proportion proposed in [11,13], an aforementioned auxiliary timer is introduced in the construction of the hybrid system to manage the non-ISS flows.

This paper is structured as follows. In Section 2, we introduce some mathematical preliminaries. Our main result – the small-gain theorem for interconnected switched systems with both ISS and non-ISS subsystems – is presented and interpreted in Section 3, followed by a corollary discussing relaxations in the assumptions to conclude GAS when all subsystems are ISS. A detailed proof, prefaced by an introduction to hybrid systems, is provided in Section 4. Section 5 concludes the paper with a short summary and an outlook on future research.

2. Preliminaries

Consider a family of dynamic systems

$$\dot{x} = f_p(x, u), \quad p \in \mathcal{P} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input and $\mathcal{P}$ is the *index set* (which can in principle be arbitrary). For all $p \in \mathcal{P}, f_p$ is locally Lipschitz and $f_p(0,0) = 0$. Given the family (1), a *switched system*

$$\dot{x} = f_\sigma(x, u) \tag{2}$$

is generated by a *switching signal* $\sigma: \mathbb{R}_{\geq 0} \to \mathcal{P}$ which specifies the index of the active system at time t. The switching signal σ is assumed to be piecewise constant and right-continuous. Let ψ_k ($k \in \mathbb{Z}_{>0}$) denote the time when the k-th switch occurs and define $\Psi := \{\psi_k : k \in \mathbb{Z}_{>0}\}$ as the set of switching time instants, which is assumed to contain no accumulation points. (Thus the switched system (2) has at most one switch at any time instant and finitely many switches in any finite time interval.) A function u is an admissible input to the switched system (2) if it is measurable and locally essentially bounded.

Following Morse [9], we say that a switching signal σ satisfies the *dwell-time condition* if there exists a $\tau_d \in \mathbb{R}_{>0}$, called the *dwell-time*, such that for all consecutive switching time instants $\psi_k, \psi_{k+1} \in \Psi$,

$$\psi_{k+1} - \psi_k \geq \tau_d. \tag{3}$$

A generalized concept was introduced by Hespanha and Morse [10]: a switching signal σ is said to satisfy the *average dwell-time condition* if there exists a $\tau_a \in \mathbb{R}_{>0}$, called the *average dwell-time*, and an $N_0 \in \mathbb{Z}_{\geq 0}$ such that

$$N(t_2, t_1) \leq N_0 + \frac{t_2 - t_1}{\tau_a} \quad \forall\, t_2 \geq t_1 \geq 0, \tag{4}$$

where $N(t_2, t_1)$ denotes the number of switchings in the time interval $(t_1, t_2]$. Note that the dwell-time condition can be interpreted as a special case of the average dwell-time condition with $N_0 = 1$ and $\tau_a = \tau_d$.

For two vectors x_1, x_2, (x_1, x_2) is used to denote their concatenation, that is, $(x_1, x_2) := (x_1^\top, x_2^\top)^\top$. For a vector $x \in \mathbb{R}^n$, we use $|x|$ to denote its Euclidean norm. For a compact set $A \subset \mathbb{R}^n$, we use $|x|_A$ to denote the Euclidean distance from a vector x to A. For a function $u: \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $\|u\|_t$ is used to denote its essential supremum (Euclidean) norm on the interval $[0, t]$.

A function $\alpha: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{K}$ if it is continuous, strictly increasing and positive definite. A function γ is of *class* $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\lim_{r\to\infty} \gamma(r) = \infty$. In particular, this implies that γ is globally invertible. A function $\beta: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for all fixed t, $\beta(r, \cdot)$ is decreasing and $\lim_{t\to\infty} \beta(r, t) = 0$ for all fixed r.

As introduced by Sontag [4], a dynamic system from family (1) is called *input-to-state stable* (ISS) if there exist functions $\gamma \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$ such that for all initial states $x(0) \in \mathbb{R}^n$ and all inputs $u: \mathbb{R}_{\geq 0} \to \mathbb{R}^m$,

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|u\|_t) \quad \forall\, t \in \mathbb{R}_{\geq 0}. \tag{5}$$

The definition of input-to-state stability (ISS) also applies to switched systems. Note that for an autonomous dynamic system (i.e. $u \equiv 0$), the ISS property (5) is equivalent to the notion of *global asymptotic stability* (GAS) [23, Proposition 2.5].

3. Main result

3.1. Interconnected switched system with both ISS and non-ISS subsystems

Consider two switched systems

$$\begin{aligned} \dot{x}_1 &= f_{1,\sigma_1}(x_1, u_1), \\ \dot{x}_2 &= f_{2,\sigma_2}(x_2, u_2), \end{aligned} \tag{6}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, and $\sigma_i \in \mathcal{P}_i$ for all $i \in \{1, 2\}$.[1] Suppose the two switched systems fulfill the same assumptions as those imposed on the switched system (2) in Section 2. If $m_1 = n_2$ and $m_2 = n_1$, an *interconnected switched system* with the state $(x_1, x_2) \in \mathbb{R}^{n_1+n_2}$ can be constructed by letting $u_1 = x_2$ and $u_2 = x_1$:

[1] We use f_{i,σ_i} instead of f_{σ_i} to avoid confusion in case the two index sets $\mathcal{P}_1, \mathcal{P}_2$ contain common elements.