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Positive nonlinear realizations of response maps

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ABSTRACT

The problem of realization of a positive response map as a positive initialized system on a finite dimensional Euclidean state space is investigated. The dynamical part of the system is described by a delta differential equation on an arbitrary time scale. This incorporates continuous- and discrete-time systems. The main result states necessary and sufficient conditions for existence of positive realization of a particular class. The criterion is expressed in the language of skew differential global universes.

1. Introduction

In systems that appear in biology, chemistry or economics the variables take often only positive or nonnegative values. Examples of such systems can be found in [1,2], where also a theory of linear positive systems was developed. One of the problems studied in these books is the problem of positive realization: passing from input–output data to a positive linear system. Many authors contributed to different solutions of this problem (see e.g. [3–8]). Realizations of positive nonlinear systems have hardly been studied. The problem of realization for the class of positive rational systems is mentioned in [9], but no solution of the problem is provided.

We study here positive realizations of positive response maps. The approach is close to that of [10,11], where polynomial, analytic and smooth realizations were considered. Similar methods have been used in [12–14] for rational and Nash systems. All the authors rely on algebraic concepts first used by E. Sontag in [15] for polynomial discrete-time systems, but none of the papers concerns positive systems.

In [11] the realization problem was studied for the class of nonlinear systems on time scales. The theory of such systems unifies the theories of continuous- and discrete-time systems. Delta differential equations, which describe the dynamics of the system, unify differential and difference equations. Actually, they include much more: dynamic equations of q-calculus, equations that arise during nonuniform sampling, systems with mixed time—partly continuous and partly discrete. We follow the setting of [11]: the systems that realize the response maps are systems on time scales. The main reason for such a general approach is saving reader's time. The result is the same for all time scales. Though the calculations for specific time scales may be different, the criteria of positive realizability and the construction of a realization are common for all time scales.

Theory of dynamical systems on time scales was laid out in [16]. Special attention was paid to linear delta differential equations. The interest in control systems on time scales dates back to 2004. The first results have concerned controllability, observability and realizations of linear constant-coefficient and varying-coefficient control systems with outputs (see e.g. [17]). Positive linear systems on time scales have been studied in [18,19,8]. There are also many developments in theory of nonlinear control systems on time scales. In [20,21] a different realization problem was solved. It consisted in passing from an input–output delta differential equation of higher order to a state-space delta differential equation with an output relation. However no positivity has been addressed.

One of the main tools we use in the paper is global universe. This is a simplified version of the concept of universe introduced by J. Johnson [22]. A global universe is a generalization of algebra. In algebra we can substitute elements into polynomials of several variables; in a global universe we can substitute elements into polynomial, analytic or smooth functions of several variables, depending on the class of the global universe. Another concept, built upon the notion of global universe, is skew differential universe. This is an extension of the concept of differential algebra.

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We construct the observation universe of a response map, which is shown to be a skew differential universe with respect to certain skew derivations. We show relations between this observation universe and the observation universe of the system that realizes the response map. The main result says that the positive response map has a positive realization (of a particular class) if and only if the observation universe of the response map is contained in some skew differential universe, with finitely many nonnegative generators.

2. Preliminaries

By $\mathbb{R}$ we shall denote the set of all real numbers, by $\mathbb{Z}$ the set of integers, and by $\mathbb{N}$ the set of natural numbers (without 0). We shall also need the set of nonnegative real numbers, denoted by $\mathbb{R}_+$ and the set of nonnegative integers $\mathbb{Z}_+$, i.e. $\mathbb{N} \cup \{0\}$. Similarly, $\mathbb{R}^k_+$ will mean the set of all column vectors in $\mathbb{R}^k$ with nonnegative components.

Let $k \in \mathbb{N}$ and let $\mathfrak{K}_k$ denote one of the following families of real functions defined on $\mathbb{R}^k$: affine, polynomial, rational, Nash (see e.g. [14]), analytic, smooth (i.e. C^∞). Let $\mathfrak{K}$ be the disjoint union of all $\mathfrak{K}_k$ for $k \in \mathbb{N}$.

Calculus on time scales is a generalization of the standard differential calculus and the calculus of finite differences. It was developed by Stefan Hilger in his Ph.D. Thesis [23]. We present here the basic definitions and facts. More information can be found e.g. in [16].

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. In particular $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = h\mathbb{Z}$ for $h > 0$ and $\mathbb{T} = q^{\mathbb{N}} := \{q^k, k \in \mathbb{N}\}$ for $q > 1$ are time scales. We assume that $\mathbb{T}$ is a topological space with the relative topology induced from $\mathbb{R}$. If $t_0, t_1 \in \mathbb{T}$, then $[t_0, t_1]_{\mathbb{T}}$ denotes the intersection of the ordinary closed interval with $\mathbb{T}$. Similar notation is used for open, half-open or infinite intervals.

For $t \in \mathbb{T}$ we define: the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ if $t \neq \sup \mathbb{T}$ and $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$ when $\sup \mathbb{T}$ is finite; the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ if $t \neq \inf \mathbb{T}$ and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$ when $\inf \mathbb{T}$ is finite; the *forward graininess function* $\mu : \mathbb{T} \to [0, \infty)$ by $\mu(t) := \sigma(t) - t$; the *backward graininess function* $\nu : \mathbb{T} \to [0, \infty)$ by $\nu(t) := t - \rho(t)$.

If $\sigma(t) > t$, then t is called *right-scattered*, while if $\rho(t) < t$, it is called *left-scattered*. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$ then t is called *right-dense*. If $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is *left-dense*.

The time scale $\mathbb{T}$ is *homogeneous*, if μ and ν are constant functions. When $\mu \equiv 0$ and $\nu \equiv 0$, then $\mathbb{T} = \mathbb{R}$ or $\mathbb{T}$ is a closed interval (in particular a half-line). When μ and ν are constant and greater than 0, then $\mu = \nu$ and $\mathbb{T} = \mu\mathbb{Z} + a$ for $a \in \mathbb{R}$.

Let us assume that $\mathbb{T}$ is *forward infinite*, i.e. for every $t \in \mathbb{T}$ there are infinitely many points in $\mathbb{T}$ that are greater than t.

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$. The *delta derivative of f at t*, denoted by $f^\Delta(t)$ or $(\frac{\Delta}{\Delta t}f)(t)$, is the real number with the property that given any ε there is a neighborhood $U = (t - \delta, t + \delta)_{\mathbb{T}}$ such that

$$|(f(\sigma(t)) - f(s)) - f^\Delta(t)(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s|$$

for all $s \in U$. If $f^\Delta(t)$ exists, then we say that f *is delta differentiable at t*. Moreover, we say that f is *delta differentiable* on $\mathbb{T}$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}$.

Example 2.1. If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = f'(t)$. If $\mathbb{T} = c\mathbb{Z}$, then $f^\Delta(t) = \frac{f(t+c)-f(t)}{c}$. If $\mathbb{T} = q^{\mathbb{N}}$, then $f^\Delta(t) = \frac{f(qt)-f(t)}{(q-1)t}$.

For a function $f : \mathbb{T} \to \mathbb{R}$ let $f^\sigma := f \circ \sigma$.

Proposition 2.2. *If $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable, then $f^\sigma = f + \mu f^\Delta$.*

Here is the chain rule on time scales.

Proposition 2.3. *Let $n \in \mathbb{N}$, $F : \mathbb{R}^n \to \mathbb{R}$ be of class C^1 and $f_1, \ldots, f_n$ be delta differentiable functions on $\mathbb{T}$. Then*

$$F(f_1, \ldots, f_n)^\Delta(t) = \int_0^1 \sum_{k=1}^n \frac{\partial F}{\partial x_k}(f_1(t) + s\mu(t)f_1^\Delta(t), \ldots, f_n(t) + s\mu(t)f_n^\Delta(t))f_k^\Delta(t)ds.$$

Corollary 2.4. *For delta differentiable functions f and g*

$$(fg)^\Delta = f^\sigma g^\Delta + f^\Delta g = fg^\Delta + f^\Delta g^\sigma = fg^\Delta + f^\Delta g + \mu f^\Delta g^\Delta.$$

Let now $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$. Consider the delta differential equation

$$x^\Delta(t) = f(t, x(t)). \tag{1}$$

A *solution* to (1) is a function x defined on some interval $[a, b) \subseteq \mathbb{T}$ and satisfying (1). If f is continuous and is of class C^1 with respect to x (the second variable), then for every initial condition $x(t_0) = x_0$ there exists a unique forward solution defined of some interval $[t_0, t_1)$ [16].

3. Positive control systems

Let Ω be an arbitrary set. It will be the set of control values. Consider a control system with output on a time scale $\mathbb{T}$

$$\Sigma : x^\Delta(t) = f(x(t), u(t)), \qquad y(t) = h(x(t)) \tag{2}$$

where $t \in \mathbb{T}$, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$ and $u(t) \in \Omega$.

Control $u : [T_0^u, T_1^u)_{\mathbb{T}} \to \Omega$ is a piecewise constant function of time, $T_0^u, T_1^u \in \mathbb{T}$, $T_0^u < T_1^u$. We shall assume that u is continuous from the right, so u is obtained by concatenation of constant controls defined on half-open intervals of the form $[a, b)_{\mathbb{T}}$. Such a control will be called *nonempty*.

We shall also need the *empty control at time t_0*, denoted by $\emptyset_{t_0}$, for any $t_0 \in \mathbb{T}$. No value is assigned to such control. Let $PC(\Omega)$ denote the set of all piecewise constant controls and all empty controls.

In [11] a more complicated definition of piecewise constant controls on a time scale was used. We believe that the current one is simpler and more natural. But to preserve simplicity we have to assume that the control is defined on a half-open interval. Concatenation of such controls is defined in an obvious way and results in a control of the same type.

For $\omega \in \Omega$ we define $f_\omega : \mathbb{R}^n \to \mathbb{R}^n$ by $f_\omega(x) := f(x, \omega)$. We shall assume that the components of f_ω for every $\omega \in \Omega$ and of h belong to $\mathfrak{K}_n$ (i.e. are affine, polynomial, rational, Nash, analytic or smooth). We shall say then that the system Σ is of class $\mathfrak{K}$. For simplicity we consider only globally defined systems, with f_ω and h defined on the entire $\mathbb{R}^n$, but without much change also partially defined systems as in [11] may be accommodated. The reason to study several classes of systems at once is the same as the reason to use time scales—to save the precious time of the reader.

Definition 3.1. A control $u : [T_0^u, T_1^u)_{\mathbb{T}} \to \Omega$ is *admissible for Σ and an initial state* $x_0 \in \mathbb{R}^n$ if there is a unique solution $x : [T_0^u, T_1^u]_{\mathbb{T}} \to \mathbb{R}^n$ of $x^\Delta(t) = f(x(t), u(t))$ corresponding to x_0 and u. Empty control $\emptyset_{t_0}$ is always admissible. The solution x of $x^\Delta(t) = f(x(t), u(t))$ corresponding to x_0 and $\emptyset_{t_0}$ is defined only at $t_0 : x(t_0) = x_0$. We shall write $x(t) = x(t, x_0, u)$ to stress dependence of the solution on the initial state and the control.

Remark 3.2. One could expect that the corresponding solution x should be defined on the same interval as u, i.e. $[T_0^u, T_1^u)$. But we shall need to evaluate x at T_1^u. If T_1^u is left-scattered, then the values