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A computational method for solving time-delay optimal control problems with free terminal time

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ABSTRACT

This paper considers a class of optimal control problems for general nonlinear time-delay systems with free terminal time. We first show that for this class of problems, the well-known time-scaling transformation for mapping the free time horizon into a fixed time interval yields a new time-delay system in which the time delays are variable. Then, we introduce a control parameterization scheme to approximate the control variables in the new system by piecewise-constant functions. This yields an approximate finite-dimensional optimization problem with three types of decision variables: the control heights, the control switching times, and the terminal time in the original system (which influences the variable time delays in the new system). We develop a gradient-based optimization approach for solving this approximate problem. Simulation results are also provided to demonstrate the effectiveness of the proposed approach.

1. Introduction

The terminal time in an optimal control problem can be either fixed or variable. For optimal control problems with variable terminal time, there are two main categories: problems in which the terminal time is a free decision parameter [1,2]; and problems in which the terminal time varies according to some stopping criterion [3–5]. This paper is concerned with problems in the first category.

For such problems, a well-known time-scaling transformation is available for mapping the free terminal time into a fixed time point [3,6]. This transformation is very useful because it converts an optimal control problem with free terminal time into a standard optimal control problem with fixed terminal time, which can (in principle) be solved using conventional methods. However, when applied to time-delay optimal control problems, the time-scaling transformation results in unexpected difficulties. In particular, as we will show in this paper, applying the time-scaling transformation to a time-delay system yields a new system in which the time delays are variable and actually depend on the terminal time. Consequently, time-delay optimal control problems with free terminal time are a major computational challenge.

We are only aware of one reference (Ref. [7]) that tackles this class of problems. This reference describes a two-stage optimization approach in which the terminal time is optimized in the outer stage, and the control function is optimized in the inner stage. The advantage of this approach is that the inner stage only requires solving optimal control problems with fixed terminal time—a task that can be readily implemented using existing numerical techniques such as control parameterization [8] or state discretization [9]. However, this approach also has two disadvantages: (i) a sequence of optimal control problems must be solved, not just one; and (ii) the time delays in the governing dynamic system must be commensurate with each other.

The purpose of this paper is to develop an alternative method that does not suffer from these limitations. Our approach involves applying the control parameterization method, in which the control is approximated by a piecewise-constant function, to the equivalent problem obtained via the time-scaling transformation. This yields an approximate finite-dimensional optimization problem, whose decision variables are the heights for the approximate control, the switching times for the approximate control and the terminal time in the original system (which influences the variable time delays in the transformed system). The main contribution of this paper is an algorithm for computing the gradients of

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the cost and constraint functionals with respect to these decision variables. By exploiting this algorithm, the approximate problem can be solved efficiently using gradient-based optimization techniques. We conclude the paper by validating this approach on a numerical example involving the harvesting of a renewable resource.

2. Problem formulation

2.1. Optimal control problem

Consider the following nonlinear control system with m time delays:

$$\dot{x}(t) = f(t, x(t-\alpha_0), \ldots, x(t-\alpha_m), u(t)), \quad t \in [0, T], \tag{1a}$$

$$x(t) = \phi(t), \quad t \leq 0, \tag{1b}$$

where $x(t) \in R^n$ is the *state vector*; $u(t) \in R^r$ is the *control vector*; $T > 0$ is a *free terminal time*; $\alpha_0 = 0$ and $\alpha_i > 0$, $i = 1, \ldots, m$, are given *time delays*; and $f : R \times R^{(m+1)n} \times R^r \to R^n$ and $\phi : R \to R^n$ are given functions.

The terminal time T in system (1a) is a free decision variable. Define

$$\mathcal{T} := \{\gamma \in R : T_{\min} \leq \gamma \leq T_{\max}\}, \tag{2}$$

where $T_{\min}$ and $T_{\max}$ are the lower and upper bounds for the terminal time, respectively. Any $T \in \mathcal{T}$ is called an *admissible terminal time*.

Furthermore, define

$$U := \{v \in R^r : a_j \leq v_j \leq b_j, j = 1, \ldots, r\}, \tag{3}$$

where a_j and b_j, $j = 1, \ldots, r$, are the lower and upper bounds for the jth control variable, respectively. Any measurable function $u : [0, \infty) \to R^r$ such that $u(t) \in U$ for almost all $t \geq 0$ is called an *admissible control*. Let $\mathcal{U}$ denote the class of all such admissible controls. Accordingly, any pair $(T, u) \in \mathcal{T} \times \mathcal{U}$ is called an *admissible pair* for system (1).

We assume throughout this paper that the following conditions are satisfied.

Assumption 1. The functions f and ϕ are continuously differentiable.

Assumption 2. There exists a real number $L > 0$ such that for all $\eta \in [0, T_{\max}]$, $y^i \in R^n$, $i = 0, \ldots, m$, and $v \in U$,

$$|f(\eta, y^0, \ldots, y^m, v)| \leq L(1 + |y^0| + \cdots + |y^m|),$$

where $|\cdot|$ denotes the Euclidean norm.

Assumptions 1 and 2 ensure that system (1) has a unique solution $x(\cdot|u)$ corresponding to each $u \in \mathcal{U}$ [10]. This solution is called the *state trajectory*.

We suppose that system (1) is subject to the following canonical constraints:

$$g_k(T, u) = \Phi_k(T, x(T|u)) \begin{cases} = 0, & k \in \mathcal{E}, \\ \geq 0, & k \in \mathcal{I}, \end{cases} \tag{4}$$

where $\mathcal{E}$ is the index set for the equality constraints; $\mathcal{I}$ is the index set for the inequality constraints; and $\Phi_k : R \times R^n \to R$, $k \in \mathcal{E} \cup \mathcal{I}$, are given continuously differentiable functions. Note that we can easily transform integral constraints into the form of (4) by introducing additional state variables. For example, consider the following integral term:

$$\int_0^T \mathcal{L}(t, x(t-\alpha_0), \ldots, x(t-\alpha_m), u(t))dt,$$

where $\mathcal{L} : R \times R^{(m+1)n} \times R^r \to R$ is a given continuously differentiable function. Clearly, this term can be replaced by $x_{n+1}(T)$, where x_{n+1} is a new state variable satisfying the dynamics

$$\dot{x}_{n+1}(t) = \mathcal{L}(t, x(t-\alpha_0), \ldots, x(t-\alpha_m), u(t)), \quad t \in [0, T],$$
$$x_{n+1}(t) = 0, \quad t \leq 0.$$

Thus, there is no loss of generality in ignoring integral terms in the constraint functions (4).

We now state our optimal control problem as follows.

Problem (P). Find an admissible pair $(T, u) \in \mathcal{T} \times \mathcal{U}$ such that the cost functional

$$g_0(T, u) = \Phi_0(T, x(T|u)) \tag{5}$$

is minimized subject to the canonical constraints (4), where $\Phi_0 : R \times R^n \to R$ is a given continuously differentiable function.

2.2. Problem transformation

Problem (P) is difficult to solve numerically because the time-delay system (1) must be integrated over a variable time horizon. For non-delay systems, this difficulty can be overcome by applying the following time-scaling transformation to map the variable interval $[0, T]$ into the fixed interval $[0, 1]$:

$$t = t(s) = Ts, \tag{6}$$

where $s \in [0, 1]$ is a new time variable. Clearly, $s = 0$ corresponds to $t = 0$, and $s = 1$ corresponds to $t = T$. This transformation is well known in the optimal control of non-delay systems. We now investigate its use for the time-delay system (1). Let $\tilde{x}(s) = x(t(s))$ and $\tilde{u}(s) = u(t(s))$. Then

$$\begin{aligned}\dot{\tilde{x}}(s) &= \frac{d}{ds}\{x(t(s))\} = \frac{dx(t(s))}{dt}\frac{dt(s)}{ds} \\ &= Tf(Ts, x(Ts-\alpha_0), \ldots, x(Ts-\alpha_m), u(Ts)) \\ &= Tf(Ts, \tilde{x}(s-\alpha_0 T^{-1}), \ldots, \tilde{x}(s-\alpha_m T^{-1}), \tilde{u}(s)).\end{aligned}$$

The initial condition (1b) becomes

$$\tilde{x}(s) = \phi(Ts), \quad s \leq 0.$$

Thus, the original control system (1) can be transformed into the following form:

$$\begin{aligned}\dot{\tilde{x}}(s) \;=\; & Tf(Ts, \tilde{x}(s-\alpha_0 T^{-1}) \\ & , \ldots, \tilde{x}(s-\alpha_m T^{-1}), \tilde{u}(s)), \quad s \in [0, 1],\end{aligned} \tag{7a}$$

$$\tilde{x}(s) = \phi(Ts), \quad s \leq 0. \tag{7b}$$

Let $\tilde{\mathcal{U}}$ be the class of all measurable functions $\tilde{u} : [0, 1] \to R^r$ such that $\tilde{u}(s) \in U$ for almost all $s \in [0, 1]$. Any pair $(T, \tilde{u}) \in \mathcal{T} \times \tilde{\mathcal{U}}$ is called an *admissible pair*. Let $\tilde{x}(\cdot|T, \tilde{u})$ denote the solution of system (7) corresponding to a given admissible pair $(T, \tilde{u}) \in \mathcal{T} \times \tilde{\mathcal{U}}$. Then the canonical constraints (4) become

$$\tilde{g}_k(T, \tilde{u}) = \Phi_k(T, \tilde{x}(1|T, \tilde{u})) \begin{cases} = 0, & k \in \mathcal{E}, \\ \geq 0, & k \in \mathcal{I}. \end{cases} \tag{8}$$

Thus, Problem (P) is equivalent to the following optimal control problem with fixed terminal time.

Problem (EP). Find an admissible pair $(T, \tilde{u}) \in \mathcal{T} \times \tilde{\mathcal{U}}$ such that the cost functional

$$\tilde{g}_0(T, \tilde{u}) = \Phi_0(T, \tilde{x}(1|T, \tilde{u})) \tag{9}$$

is minimized subject to the canonical constraints (8).