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# Nilpotent semigroups for the characterization of flat outputs of switched linear and LPV discrete-time systems

Jérémy Parriaux, Gilles Millérioux *

*Université de Lorraine, Research Center for Automatic Control of Nancy (CRAN UMR 7039), 2 rue Jean Lamour, 54519 Vandoeuvre-les-Nancy, France*

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ABSTRACT

This paper addresses the problem of flat output characterization for switched linear systems. The characterization is also extended to LPV systems. The characterization is based on the notion of nilpotent semigroups. A complete corresponding recursive algorithm is provided. It stops after a finite number of steps bounded by the dimension of the system. Illustrative examples, for the respective class of switched linear and LPV systems, highlight the efficiency of the characterization.



## 1. Introduction

Flatness is a control-theoretical concept introduced in [1] and the assets of flatness-based approaches are well-established. A deep insight on flatness along with applications can be found in the book [2]. For a flat discrete-time system (linear or nonlinear), the state variable as well as the input of the system can be written as some function of the output (including forward and backward shifts in the output). Such a property is especially interesting both for state reconstruction and control perspectives. Indeed, it is clear from the definition that flatness provides a generic way of reconstructing the state vector despite possibly unknown inputs. Even more is true, flatness is a structural property of a dynamical system and so ensures the existence of an unknown input observer without any *a priori* structure of the observer. For control purposes, flatness is also relevant insofar as, given a flat output, the definition of flatness provides in a straightforward manner a constructive way of designing a feedforward control to track a prescribed trajectory of the plant output. This being the case, an important issue related to flatness is the problem of checking whether a given output of a dynamical system is flat or not. Indeed, it is precisely the flat output which will be exclusively used for the design of the controller or the state reconstructor according to the purpose. A first approach consists in trying to directly agree with the definition, that is attempting to express the input and the state vector as a function exclusively involving derivatives of the output in the continuous case or shifts of the output in the discrete-time case. A more relevant approach has been proposed in [3] for continuous linear systems. For nonlinear systems, flat output characterization has been addressed to a much lesser extent. We may refer to the recent work [4] dealing with flatness of time invariant nonlinear discrete-time systems from a behavioral perspective. In this paper, we propose a characterization for switched linear systems. It is based on the notion of nilpotent semigroups and a complete tractable algorithm is given for checking the conditions. Furthermore, it is shown that the theoretical condition and the corresponding algorithm can be extended to LPV systems with little effort. The layout is as follows. In Section 2, we recall some basics on flatness with a special emphasis on switched linear systems. Section 3 is devoted to flat output characterization and a description of the corresponding algorithm. An extension to LPV systems is proposed in Section 4. Finally, Section 5 is devoted to illustrative examples.

## 2. Preliminaries and definitions

Throughout this paper, we shall examine switched linear systems in the form

$$\begin{cases} x_{k+1} = A_{\sigma(k)}x_k + B_{\sigma(k)}u_k \\ y_k = C_{\sigma(k)}x_k + D_{\sigma(k)}u_k. \end{cases} \quad (1)$$

The state vector is $x_k \in \mathbb{R}^n$, the input is $u_k \in \mathbb{R}^m$ and the output is $y_k \in \mathbb{R}^p$. All the matrices, namely $A_{\sigma(k)} \in \mathbb{R}^{n\times n}$, $B_{\sigma(k)} \in \mathbb{R}^{n\times m}$, $C_{\sigma(k)} \in \mathbb{R}^{p\times n}$ and $D_{\sigma(k)} \in \mathbb{R}^{p\times m}$ belong to the respective finite

* Corresponding author. Tel.: +33 0 3 83 68 51 37.
*E-mail addresses:* jeremy.parriaux@univ-lorraine.fr (J. Parriaux), gilles.millerioux@univ-lorraine.fr (G. Millérioux).

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sets of cardinality $J$: $\mathcal{A} = \{A_1, \dots, A_J\}$, $\mathcal{B} = \{B_1, \dots, B_J\}$, $\mathcal{C} = \{C_1, \dots, C_J\}$ and $\mathcal{D} = \{D_1, \dots, D_J\}$. At a given time $k$, the mode is delivered by a switching function $\sigma : k \in \mathbb{N} \longmapsto \sigma(k) \in \{1, \dots, J\} = \mathcal{J}$. A sequence of modes (also called a path) over an interval of time $[k_1, k_2]$, that is $\{\sigma(k_1), \dots, \sigma(k_2)\}$, is denoted by $\{\sigma\}_{k_1:k_2}$. For a given switching rule $\sigma$, the set of corresponding mode sequences over an interval of time $[k, k+T]$ belongs to $\mathcal{J}^{T+1}$. Let $\mathcal{U}$ be the space of input sequences over the time interval $[0, \infty)$ and $\mathcal{Y}$ the corresponding output space. At time $k$, for each initial state $x_k \in \mathbb{R}^n$, when the system (1) is driven by the input sequence $\{u\}_{k:k+T} = \{u_k, \dots, u_{k+T}\} \in \mathcal{U}$, for a mode sequence $\{\sigma\}_{k:k+T}$, $\{x(x_k, \sigma, u)\}_{k:k+T}$ refers to the solution of (1) in the interval of time $[k, k+T]$ starting from $x_k$ and $\{y(x_k, \sigma, u)\}_{k:k+T} \in \mathcal{Y}$ refers to the corresponding output sequence in the same interval of time $[k, k+T]$.

For any integer $n$, $\mathbf{1}_n$ refers to the $n$-dimensional identity matrix and $\mathbf{0}_{n\times m}$ stands for the $n \times m$ zero matrix. If irrelevant, the dimension of the zero matrix is omitted and it shall be merely written as $\mathbf{0}$. We introduce the subsequent vectors and matrices:

$$u_{k:k+i} = \begin{pmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+i} \end{pmatrix}, \qquad y_{k:k+i} = \begin{pmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+i} \end{pmatrix} \tag{2}$$

$$I_{m\times r} = \begin{pmatrix} \mathbf{1}_m & \mathbf{0}_{m\times(m\cdot r)} \end{pmatrix}$$

$$\mathcal{O}_{\sigma(k:k+i)} = \begin{pmatrix} C_{\sigma(k)} \\ C_{\sigma(k+1)}A_{\sigma(k)} \\ \vdots \\ C_{\sigma(k+i)}A_{\sigma(k)}^{\sigma(k+i-1)} \end{pmatrix}. \tag{3}$$

The matrix $\mathcal{O}_{\sigma(k:k+i)}$ involves the transition matrix defined by

$$A_{\sigma(k_0)}^{\sigma(k_1)} = A_{\sigma(k_1)}A_{\sigma(k_1-1)} \cdots A_{\sigma(k_0)} \quad \text{if } k_1 \geq k_0$$
$$= \mathbf{1}_n \quad \text{if } k_1 < k_0.$$

Finally, we recursively define the matrix

$$M_{\sigma(k:k+i)} = \begin{pmatrix} D_{\sigma(k)} & \mathbf{0} \\ \mathcal{O}_{\sigma(k:k+i)}B_{\sigma(k)} & M_{\sigma(k+1:k+i)} \end{pmatrix} \tag{4}$$

with

$$M_{\sigma(k:k)} = D_{\sigma(k)}.$$

Let us notice that the notation $\sigma(k : k+i)$, which points out that the related matrix depends on the sequence $\{\sigma(k), \dots, \sigma(k+i)\}$ is somehow abusive since $\sigma$ is defined over $\mathbb{N}$ and not over $\mathbb{N}^{i+1}$. However, since it does not induce confusion, such a notation will be used accordingly for the sake of shortness.

Flatness is closely related to the notions of left invertibility which actually stands for a necessary condition. Roughly speaking, invertibility of a dynamical system is the ability of uniquely determining the input sequence from the output sequence. The works dealing with left invertibility reported in [5] are considered throughout the literature as the pioneering ones. Left invertibility for switched linear systems has been addressed in [6] for continuous-time systems and in [7,8] for discrete-time systems.

The concept of left inverse systems, related to left invertibility, will play a central role for our purpose. The following definition is in accordance with the papers [7–9].

**Definition 1.** A system, with state vector $\widehat{x}_k$, is a left $r$-delay inverse for (1) if, under identical initial conditions $x_0$ and identical sequences $\{\sigma\}_{0:\infty}$, there exists a non negative integer $r$ such that, when driven by $y_{k:k+r}$, the equalities $\widehat{x}_{k+r} = x_k$ and $\widehat{u}_{k+r} = u_k$ for all $k \geq 0$ are ensured, $\widehat{u}_k$ being the output of (1) at time $k$. The non negative integer $r$ is called the inherent delay.

Let us notice that the terminology of $r$-delay inverse and inherent delay is borrowed from the work [10] which deals with linear systems. Besides, the consideration of the initial condition $x_0$ stands as a counterpart of the continuous case and the definition of *invertibility at point* $x_0$ introduced in [11]. Actually, the initial condition $x_0$ has been disregarded in [10] by assuming that it is zero or that "its effect can be subtracted".

The papers [7–9] give an explicit form of the left $r$-delay inverse system for (1). It is recalled below.

$$\begin{cases} \widehat{x}_{k+r+1} = P_{\sigma(k:k+r)}\widehat{x}_{k+r} + B_{\sigma(k)}I_{m\times r}(M_{\sigma(k:k+r)})^{\dagger}y_{k:k+r} \\ \widehat{u}_{k+r} = -I_{m\times r}(M_{\sigma(k:k+r)})^{\dagger}\mathcal{O}_{\sigma(k:k+r)}\widehat{x}_{k+r} \\ \quad + I_{m\times r}(M_{\sigma(k:k+r)})^{\dagger}y_{k:k+r} \end{cases} \tag{5}$$

with

$$P_{\sigma(k:k+r)} = A_{\sigma(k)} - B_{\sigma(k)}I_{m\times r}(M_{\sigma(k:k+r)})^{\dagger}\mathcal{O}_{\sigma(k:k+r)}. \tag{6}$$

The matrix $(M_{\sigma(k:k+r)})^{\dagger}$ is the classical Moore–Penrose generalized inverse of $M_{\sigma(k:k+r)}$. The matrices $P_{\sigma(k:k+r)}$ are called left inverse dynamical matrices.

### 2.1. Flatness

**Definition 2** ([9]). A square ($p = m$) dynamical system is said to be *flat* if there exists a set of independent variables $y_k$, referred to as flat outputs, such that all system variables can be expressed as a function of the flat output and a finite number of its backward and/or forward shifts. In particular, there exist two functions $\mathcal{F}$ and $\mathcal{G}$ which obey

$$\begin{cases} x_k = \mathcal{F}(y_{k+k_{\mathcal{F}}}, \dots, y_{k+k'_{\mathcal{F}}}) \\ u_k = \mathcal{G}(y_{k+k_{\mathcal{G}}}, \dots, y_{k+k'_{\mathcal{G}}}) \end{cases} \tag{7}$$

where $k_{\mathcal{F}}$, $k'_{\mathcal{F}}$, $k_{\mathcal{G}}$ and $k'_{\mathcal{G}}$ are $\mathbb{Z}$-valued integers.

The issue of flat output characterization consists in checking whether or not a given output of a dynamical system is flat. Theorem 1 stated in [9] and recalled below gives a first characterization by considering the left inverse dynamical system (5).

**Theorem 1** ([9]). *An output $y_k$ of the system (1) assumed to be square, with left inherent delay $r$, is a* flat output *if there exists a positive integer $K < \infty$ such that, for all sequences in $\mathcal{J}^{r+K}$, the following equality, involving the product of left inverse dynamical matrices, applies for all $k \geq 0$:*

$$P_{\sigma(k+K-1:k+K-1+r)}P_{\sigma(k+K-2:k+K-2+r)} \cdots P_{\sigma(k:k+r)} = \mathbf{0}. \tag{8}$$

Condition (8) only involves matrices (6) of the left $r$-delay inverse system (5). Besides, the matrices (6) depend on sequences of modes. Hence, even if $\sigma$ is arbitrary, the sequences parametrizing two successive matrices involved in the product (8) are related to each other. To cope with this constraint without introducing too heavy notations and to make the subsequent technical developments more explicit, it is convenient to define an auxiliary system and to rewrite Theorem 1 accordingly.

### 2.2. Auxiliary system

Let us define the auxiliary system of (1) as the switched linear system given by

$$q_{k+1} = Q_{\sigma'(k)}q_k \tag{9}$$

with $q_k \in \mathbb{R}^n$ and $\sigma'$ a switching rule defined as follows.

Consider the mapping $\phi : \mathcal{J}^{r+1} \to \mathcal{H} = \{1, \dots, J^{r+1}\}$ that assigns to each possible sequence $\{\sigma(k), \dots, \sigma(k+r)\}$ an integer $h$ from the set $\mathcal{H}$ which uniquely identifies the sequence.